

G-SYMMETRIC MONOIDAL CATEGORIES OF MODULES OVER EQUIVARIANT COMMUTATIVE RING SPECTRA

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ABSTRACT. We describe the multiplicative structures that arise on categories of equivariant modules over certain equivariant commutative ring spectra. Building on our previous work on \mathcal{N}_∞ ring spectra, we construct categories of equivariant operadic modules over \mathcal{N}_∞ rings that are structured by equivariant linear isometries operads. These categories of modules are endowed with equivariant symmetric monoidal structures, which amounts to the structure of an “incomplete Mackey functor in homotopical categories”. In particular, we construct internal norms which satisfy the double coset formula. We regard the work of this paper as a first step towards equivariant derived algebraic geometry.

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1. INTRODUCTION

Stable homotopy theory has been revolutionized over the last twenty years by the development of symmetric monoidal categories of spectra [7, 15, 12]. Commutative monoids in these categories model E_∞ ring spectra. Arguably the most important consequence of this machinery is the ability to have tractable point-set models for homotopical categories of modules over an E_∞ ring spectrum R . In the equivariant setting, analogous symmetric monoidal categories of G -spectra have been constructed, most notably the category of orthogonal G -spectra [14]. Once again commutative monoids model E_∞ ring spectra and so there are good point-set models for homotopical categories of modules over such an equivariant E_∞ ring spectrum.

Modules over a commutative ring orthogonal G -spectrum R form a “ G -symmetric monoidal category” [9]. Roughly speaking, for each G -set T , we have an internal norm in the category of R -modules; for an orbit G/H , the internal norm is precisely the composite of the R -relative norm ${}_R N_H^G$ and the forgetful functor from R -modules to $\iota_H^* R$ -modules. These internal norms are compatible with disjoint unions of G -sets and with restrictions to subgroups, and if the set has a trivial action and cardinality n , then we simply recover the smash power functors $X \mapsto X^n$.

However, in contrast to the non-equivariant setting, there are many possible notions of E_∞ ring spectra when working over a nontrivial finite group G . The commutative monoids in orthogonal G -spectra are just one end of the spectrum of possible multiplicative structures. In a previous paper, we described this situation in detail by explaining how such multiplications can be structured by \mathcal{N}_∞ operads [3]. Roughly speaking, just as a commutative ring is characterized by compatible multiplication maps $R^{\wedge n} \rightarrow R$ as n varies over the natural numbers, a commutative G -ring is characterized by compatible equivariant multiplication maps $R^{\wedge T} \rightarrow R$, where here T is a G -set. The \mathcal{N}_∞ operads structure which such equivariant norms exist for a given commutative ring, expressed in terms of compatible families of subgroups of $G \times \Sigma_n$. Specifically, associated to an \mathcal{N}_∞ operad \mathcal{O} there is a coefficient system $\mathcal{C}(\mathcal{O})$ which controls the “admissible” G -sets T for which equivariant multiplications exist. The commutative monoids in orthogonal G -spectra correspond to the “complete” \mathcal{N}_∞ operads which permit all norms.

In this paper, we turn to the study of the G -symmetric monoidal structure on categories of operadic modules associated to algebras over particular \mathcal{N}_∞ operads: the linear isometries operads determined by a (possibly incomplete) G -universe U . Here the admissible sets for U will play a second role; for an \mathcal{O} -algebra R , the admissible sets determine additional structure on the underlying symmetric monoidal category of R -modules. Specifically, for each admissible G -set T , we have an internal norm in the category of R -modules for an \mathcal{O} -algebra R .

In order to describe this structure, it is convenient to instead consider the collection of categories of modules over $\iota_H^* R$, where $H \subseteq G$ is a closed subgroup of G and ι_H^* is the forgetful functor. The extra structure on the category of R -modules then is encoded in functors

$$\iota_H^*: \text{Mod}_{\iota_K^* R} \longrightarrow \text{Mod}_{\iota_H^* R} \quad \text{and} \quad (\iota_K^*) N_H^K: \text{Mod}_{\iota_H^* R} \longrightarrow \text{Mod}_{\iota_K^* R}$$

for $H \subset K \subset G$ that assemble into a kind of “incomplete Mackey functor” of homotopical categories. The internal norms arise from the composite functors $N_H^G \iota_H^* H(-)$, extended to arbitrary G -sets T by decomposing T into a disjoint union

of orbits G/H and smashing together the corresponding composites. The compatibility conditions in particular express the double coset formula.

More precisely, we have the following functors:

Theorem 1.1. *Let G be a finite group and U a G -universe. Let R be an algebra in orthogonal G -spectra over \mathcal{L}_U , the linear isometries operad structured by U . Then for each $H \subset G$ there exists a symmetric monoidal model category $\mathcal{M}_{\iota_H^* R}$ of $\iota_H^* R$ -modules. For each $H \subset K \subset G$ such that K/H is an admissible K -set for U , there exist homotopical functors*

$$(\iota_K^* R) N_{H, \iota_H^* U}^{K, \iota_K^* U} : \mathcal{M}_{\iota_H^* R} \longrightarrow \mathcal{M}_{\iota_K^* R} \quad \text{and} \quad \iota_H^* : \mathcal{M}_{\iota_K^* R} \longrightarrow \mathcal{M}_{\iota_H^* R}.$$

The internal norms now arise from these functors:

Definition 1.2. Let G be a finite group and U be a G -universe. For an H -set T , writing $T = H/K_1 \amalg H/K_2 \amalg \dots \amalg H/K_m$, define

$$(\iota_H^* R) N^T : \mathcal{M}_{\iota_H^* R} \longrightarrow \mathcal{M}_{\iota_H^* R}$$

by the formula

$$(\iota_H^* R) N^T X = \left((\iota_H^* R) N_{K_1}^H \iota_{K_1}^* X \right) \wedge_{\iota_H^* R} \left((\iota_H^* R) N_{K_2}^H \iota_{K_2}^* X \right) \wedge_{\iota_H^* R} \dots \wedge_{\iota_H^* R} \left((\iota_H^* R) N_{K_m}^H \iota_{K_m}^* X \right).$$

More generally, define

$${}_R N^T : \mathcal{M}_R \longrightarrow \mathcal{M}_R$$

by the formula

$${}_R N^T X = G_+ \wedge_H \left((\iota_H^* R) N^T X \right).$$

The G -symmetric monoidal structure on \mathcal{M}_R is encoded by the following relations between the internal norms and the forgetful functors:

Theorem 1.3. *Let G be a finite group and U be a G -universe.*

- (1) *For $H_1 \subseteq H_2 \subseteq H_3 \subseteq G$, there are natural isomorphisms*

$$N_{H_2}^{H_3} N_{H_1}^{H_2} \cong N_{H_1}^{H_3} \quad \text{and} \quad \iota_{H_1}^* \iota_{H_2}^* \cong \iota_{H_1}^*$$

that descend to the derived category,

- (2) *For any H -sets T_1 and T_2 , there are natural isomorphisms $N^{T_1 \times T_2} X \cong N^{T_1} N^{T_2} X$ that descend to the derived category when T_1 and T_2 are admissible, and*

- (3) *For an admissible H -set T , the derived composite $\iota_K^* N^T M$ is naturally equivalent to $N^{\iota_K^* T} \iota_K^* M$.*

The last of these relations is a version of the double coset formula.

When $G = e$, the structure described by Theorem 1.3 is simply the usual symmetric monoidal structure on orthogonal spectra; the functors N^T for a set T are just the smash powers $X^{\wedge |T|}$. When U is the complete universe, this structure is precisely the G -symmetric monoidal structure on R -modules obtained by choosing a model of R that is a commutative monoid in orthogonal G -spectra.

Note that we have avoided trying to precisely formulate the notion of an incomplete Mackey functor of homotopical categories here, choosing instead to explicitly write out the structure and some of the coherences. However, if we are willing to pass to the homotopy category, we can state the following result.

Corollary 1.4. *Let R be an algebra in orthogonal G -spectra over \mathcal{L}_U . Let $\mathcal{B}_{G,U}$ denote the bicategory of spans of the admissible sets for \mathcal{L}_U . There exists a 2-functor from $\mathcal{B}_{G,U}$ to the 2-category of triangulated categories, exact functors, and natural isomorphisms that takes an admissible set G/H to $\mathcal{M}_{\iota_H^* R, \iota_H^* U}$.*

However, the coherences necessary for the definition of an incomplete Mackey functor at the level of homotopical categories is most easily handled using the formalism of ∞ -categories; we expect such a treatment to come from the forthcoming work of [5]. (See also [4] for a treatment of equivariant permutative categories from this kind of perspective. A different approach to equivariant permutative categories is described in [8].)

In order to explain the restriction to \mathcal{N}_∞ operads that can be modeled as linear isometries operads, we need to explain the strategy of proof for Theorem 1.1. Our approach is to adapt the strategy of EKMM to study operadic multiplications on G -spectra. Let $\mathcal{S}p_G$ denote the category of orthogonal G -spectra on a complete universe. Fix a different (possibly incomplete) G -universe U . Then there is a monad \mathbb{L}_U on $\mathcal{S}p_G$, specified by the formula

$$X \mapsto \mathcal{L}(U, U)_+ \wedge X,$$

where $\mathcal{L}(U, U)$ is the G -space of non-equivariant linear isometries from U to U (with G acting by conjugation).

The category $\mathcal{S}p_G[\mathbb{L}_U]$ of \mathbb{L}_U -algebras has a model structure that is Quillen equivalent to the standard model structures on $\mathcal{S}p_G$. Moreover, it has a new symmetric monoidal product \wedge_U such that the underlying orthogonal G -spectrum of $X \wedge_U Y$ is equivalent to $X \wedge Y$. But now monoids and commutative monoids for \wedge_U are precisely (non)-symmetric algebras for the G -linear isometries operad for U . Just as in the category of spectra, we can restrict to the unital objects in $\mathcal{S}p_G[\mathbb{L}_U]$ to obtain a symmetric monoidal category $G\mathcal{S}_U$. All of these categories can be equipped with symmetric monoidal model category structures. Using these symmetric monoidal model categories, we construct symmetric monoidal module categories for an \mathcal{N}_∞ ring R structured by the G -linear isometries operad for U .

We expect that Theorem 1.1 is true more generally for any \mathcal{N}_∞ operad, but it is difficult to obtain control on categories of operadic modules over operads other than the linear isometries operad using point-set techniques. In fact, a substantial part of the work of this paper involves verification of delicate point-set facts about the linear isometries operad that are simply not true for an arbitrary \mathcal{N}_∞ operad, just as in [7]. Unfortunately, as we explain in [3, Theorem 4.24], there are equivariant operads which arise from “little disks” constructions that are not equivalent to equivariant linear isometries operads for any universe. Again, we expect that it is more tractable to handle these sorts of homotopical questions in the ∞ -categorical setting; specifically, working with equivariant ∞ -operads structured over the nerve of distinguished subcategories of the category of finite G -sets.

Our interest in Theorem 1.1 comes in part from consequences for the foundations of equivariant derived algebraic geometry. In their study of the multiplicative properties of equivariant Bousfield localization, the second author and Hopkins showed that localization of an \mathcal{N}_∞ ring spectrum can change the universe that structures the multiplication [9]. This implies that there is not necessarily a “genuine” affine scheme associated to a commutative ring orthogonal G -spectrum when we work with the Zariski topology. Work of Nakaoka shows that something similar is true

for Tambara functors: there does not exist a sheaf of Tambara functors on the Zariski site of a Tambara functor [16].

However, by restriction of structure, every equivariant commutative ring spectrum R is also an algebra over $\mathcal{L}_{\mathbb{R}^\infty}$, the linear isometries operad for a trivial universe. Bousfield localization always preserves the property of being an algebra over $\mathcal{L}_{\mathbb{R}^\infty}$, so in particular, we do have a sheaf of such rings in the Zariski topology. Therefore, using the work of this paper we can define equivariant derived affine schemes (and then more general derived schemes by gluing) in this fashion. We intend to return to the study of equivariant derived schemes in a subsequent paper.

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2. REVIEW OF \mathcal{N}_∞ OPERADS

In this section, we review the framework for describing equivariant commutative ring spectra that we will work with in the paper. We refer the reader to [3] for a more detailed discussion.

Let G be a finite group and let $G\mathcal{S}$ denote the category of orthogonal G -spectra structured by a complete universe and with morphisms all (not necessarily equivariant) maps. We will tacitly suppress notation for the “additive” universe implicit in the definition of $G\mathcal{S}$, as we are focused on multiplicative phenomena. Recall that the category $G\mathcal{S}$ is a complete and cocomplete closed symmetric monoidal category under the smash product \wedge with unit the equivariant sphere spectrum \mathbb{S}_G . We will write $F(-, -)$ for the internal mapping G -spectrum in $G\mathcal{S}$. The category $G\mathcal{S}$ is enriched over based G -spaces and has tensors and cotensors; for X an object of $G\mathcal{S}$, the tensor with a based G -space A is given by the smash product $A \wedge X$ and the cotensor by the function spectrum $F(A, X)$.

The enrichment of $G\mathcal{S}$ means that we can regard operads in G -spaces as acting on objects of $G\mathcal{S}$ via the addition of a G -fixed disjoint basepoint and the tensor. Given a G -operad in spaces, recall the following definition from [3, Definition 3.7].

Definition 2.1. An \mathcal{N}_∞ operad is a G -operad \mathcal{O} such that

- (1) The space \mathcal{O}_0 is G -contractible,
- (2) The action of Σ_n on \mathcal{O}_n is free,
- (3) and \mathcal{O}_n is a universal space for a family $\mathcal{F}_n(\mathcal{O})$ of subgroups of $G \times \Sigma_n$ which contains all subgroups of the form $H \times \{1\}$.

For any \mathcal{N}_∞ operad \mathcal{O} , there is an associated category $\mathcal{O}\text{-Alg}$ of \mathcal{O} -algebras in $G\mathcal{S}$. We will be particularly interested in the algebras associated to the G -linear isometries operads. Fix a possibly incomplete universe U of finite-dimensional G -representations; we adopt the standard convention that U contains a trivial representation and each of its finite-dimensional subrepresentations infinitely often. We do not assume any relationship between U and the “additive” universe that arises in the definition of $G\mathcal{S}$.

Definition 2.2. The G -linear isometries operad \mathcal{L}_U has n th space $\mathcal{L}_U(n) = \mathcal{L}(U^n, U)$ the G -space of non-equivariant linear isometries $U^n \rightarrow U$ equipped with

the conjugation action. The distinguished element $1 \in \mathcal{L}_U(1)$ is the identity map and the operad structure maps are induced by composition and direct sum.

Recall from [3, Theorem 4.24] that the G -linear isometries operads do not usually describe all of the possible \mathcal{N}_∞ operads. Nonetheless, they do capture many examples of interest, in particular including the trivial and complete multiplicative universes.

One of the major themes of our previous study of \mathcal{N}_∞ operads was that the essential structure encoded by an operad \mathcal{O} is the collection of *admissible sets*. We now review the relevant definitions from [3, §3].

Definition 2.3. A *symmetric monoidal coefficient system* is a contravariant functor $\underline{\mathcal{C}}$ from the orbit category of G to the category of symmetric monoidal categories and strong symmetric monoidal functors. The *value at H* of a symmetric monoidal coefficient system $\underline{\mathcal{C}}$ is $\underline{\mathcal{C}}(G/H)$, and will often be denoted $\underline{\mathcal{C}}(H)$.

The most important example of a symmetric monoidal coefficient system for us is the coefficient system of finite G -sets.

Definition 2.4. Let $\underline{\text{Set}}$ be the symmetric monoidal coefficient system of finite sets. The value at H is Set^H , the category of finite H -sets and H -maps. The symmetric monoidal operation is disjoint union.

We will associate to every \mathcal{N}_∞ operad a subcoefficient system of $\underline{\text{Set}}$. The operadic structure gives rise to additional structure on the coefficient system.

Definition 2.5. We say that a full sub symmetric monoidal coefficient system \mathcal{F} of $\underline{\text{Set}}$ is *closed under self-induction* if whenever $H/K \in \mathcal{F}(H)$ and $T \in \mathcal{F}(K)$, $H \times_K T \in \mathcal{F}(H)$.

Definition 2.6. Let $\mathcal{C} \subset \mathcal{D}$ be a full subcategory. We say that \mathcal{C} is a *truncation subcategory* of \mathcal{D} if whenever $X \rightarrow Y$ is monic in \mathcal{D} and Y is in \mathcal{C} , then X is also in \mathcal{C} . A truncation sub coefficient system of a symmetric monoidal coefficient system $\underline{\mathcal{D}}$ is a sub coefficient system that is levelwise a truncation subcategory.

In particular, for finite G -sets, truncation subcategories are subcategories that are closed under passage to subobjects and which are closed under isomorphism.

Definition 2.7. An *indexing system* is a truncation sub symmetric monoidal coefficient system $\underline{\mathcal{F}}$ of $\underline{\text{Set}}$ that contains all trivial sets and is closed under self induction and Cartesian product.

One of the main structural theorems about \mathcal{N}_∞ operads [3, 4.17] is that an \mathcal{N}_∞ operad \mathcal{O} determines an indexing system of admissible sets. This connection arises from the standard observation that subgroups Γ of $G \times \Sigma_n$ such that $\Gamma \cap (\{1\} \times \Sigma_n) = \{1\}$ arise as the graphs of homomorphisms $H \rightarrow \Sigma_n$, for some $H \subseteq G$.

3. POINT-SET CATEGORIES OF MODULES OVER AN \mathcal{N}_∞ ALGEBRA

In this section, we describe an approach to constructing categories of modules over \mathcal{N}_∞ algebras that proceeds via a rigidification argument. Of course, for any given \mathcal{N}_∞ operad \mathcal{O} and an \mathcal{O} -algebra R in orthogonal G -spectra, we can construct a category of operadic modules over R . However, experience in the non-equivariant case teaches us that for practical work it is extremely convenient to have rigid

models of such categories that are equipped with a symmetric monoidal smash product.

Specifically, for each linear isometries operad $\mathcal{O} = \mathcal{L}(U)$, we will construct a symmetric monoidal structure on a category Quillen equivalent to orthogonal G -spectra such that monoids and commutative monoids correspond to \mathcal{O} -algebras. We describe how to produce such a structure by adapting the techniques pioneered in the development of the EKMM category of S -modules. See also [1, 2, 13, 17] for other categories in which this kind of approach has been developed. We can then define modules over an \mathcal{O} -algebra R in the evident fashion.

We are not able to rigidify algebras and modules over \mathcal{N}_∞ operads which are not equivalent to equivariant linear isometries operads. Although in these cases we can give a homotopical construction of the tensor product of operadic \mathcal{O} -modules in terms of the bar construction, we do not have good point-set control.

3.1. The point-set theory of \mathbb{L}_U -algebras in orthogonal spectra. We begin by discussing the point-set details of the category of algebras for a monad obtained from the first part of the equivariant linear isometries operad. Recall that GS denotes the category of orthogonal G -spectra structured by a complete G -universe. Since the universe implicit in the definition of GS does not play an essential role in what follows (once the weak equivalences are fixed), we continue to suppress this choice from the notation.

Remark 3.1. It is possible to carry out the work of this paper in the context of an incomplete additive universe on GS ; the simplest case arises when the additive and multiplicative universes are the same. We leave this elaboration (and its attendant complications) to the interested reader.

Fix a (possibly incomplete) universe U . Let

$$\mathcal{L}_U(1) = \mathcal{L}(U, U)$$

denote the G -space of linear isometries $U \rightarrow U$; i.e., the space of non-equivariant linear isometries $U \rightarrow U$ equipped with the conjugation action. More generally, we write $\mathcal{L}_U(n)$ to denote $\mathcal{L}(U^n, U)$, the n th space of the equivariant linear isometries operad.

Since GS is tensored over based G -spaces, the formula

$$X \mapsto \mathcal{L}_U(1)_+ \wedge X$$

specifies a monad $\mathbb{L}_U : GS \rightarrow GS$. The monadic structure maps are induced by the identity element $\text{id}_U \in \mathcal{L}_U(1)$ and the composition

$$\mathcal{L}(U, U) \times \mathcal{L}(U, U) \longrightarrow \mathcal{L}(U, U).$$

Definition 3.2. Let $GS[\mathbb{L}_U]$ denote the category of \mathbb{L}_U -algebras in GS .

Since the monad \mathbb{L}_U has a right adjoint $F(\mathcal{L}_U(1)_+, -)$, the observation of [7, I.4.3] implies that this right adjoint determines a comonad $\mathbb{L}_U^\#$ such that the category of coalgebras over $\mathbb{L}_U^\#$ is equivalent to $GS[\mathbb{L}_U]$. As a consequence we conclude the following result about the existence of limits and colimits.

Lemma 3.3. *The category $GS[\mathbb{L}_U]$ is complete and cocomplete, with limits and colimits created in GS . Similarly, $GS[\mathbb{L}_U]$ has tensors and cotensors with based G -spaces; the indexed colimits and limits are created in GS .*

The category $GS[\mathbb{L}_U]$ is equipped with mapping G -spectra $F_{GS[\mathbb{L}_U]}(-, -)$ defined by the equalizer

$$F(X, Y) \rightrightarrows F(\mathbb{L}_U X, Y),$$

where the maps are induced by the action $\mathbb{L}_U X \rightarrow X$ and the adjoint of the composite

$$(\mathbb{L}_U X) \wedge F(X, Y) \cong \mathbb{L}_U(X \wedge F(X, Y)) \longrightarrow \mathbb{L}_U Y \longrightarrow Y.$$

Next, we note that any orthogonal G -spectrum can be given a trivial $GS[\mathbb{L}_U]$ structure. Specifically, in addition to the free \mathbb{L}_U -algebra functor

$$\mathcal{L}_U(1)_+ \wedge (-): GS \longrightarrow GS[\mathbb{L}_U],$$

there is another functor $p^*: GS \rightarrow GS[\mathbb{L}_U]$ determined by the unique projection map $p: \mathcal{L}_U(1) \rightarrow *$; i.e., we can equip any orthogonal G -spectrum X with the trivial structure map

$$\mathcal{L}_U(1)_+ \wedge X \longrightarrow (*)_+ \wedge X \cong X.$$

We will be most interested in the sphere spectrum \mathbb{S}_G regarded as an \mathbb{L}_U -algebra in this fashion. The pullback functor is the right adjoint of a functor $Q: GS[\mathbb{L}_U] \rightarrow GS$ specified by the formula $QX = \mathbb{S}_G \wedge_{\Sigma_{\mathbb{L}_U(1)}} X$.

We now define a closed weak symmetric monoidal structure on $GS[\mathbb{L}_U]$ with unit \mathbb{S}_G . (Recall that a weak symmetric monoidal category has a product and a unit satisfying all of the axioms of a symmetric monoidal category except that the unit map is not required to be an isomorphism [7, II.7.1].)

Definition 3.4. Let X, Y be objects of $GS[\mathbb{L}_U]$. We define the smash product \wedge_U to be the coequalizer of the diagram

$$(\mathcal{L}_U(2) \times \mathcal{L}_U(1) \times \mathcal{L}_U(1))_+ \wedge (X \wedge Y) \rightrightarrows \mathcal{L}_U(2)_+ \wedge (X \wedge Y) \rightarrow X \wedge_U Y$$

where the maps are specified by the actions of $\mathcal{L}_U(1)_+$ on X and Y and the right action of $\mathcal{L}_U(1) \times \mathcal{L}_U(1)$ on $\mathcal{L}_U(2)$ via block sum and precomposition.

We will sometimes write this coequalizer using the notation

$$\mathcal{L}_U(2) \times_{\mathcal{L}_U(1) \times \mathcal{L}_U(1)} (X \wedge Y).$$

Here the left action of $\mathcal{L}_U(1)$ on $\mathcal{L}_U(2)$ induces a left action of $\mathcal{L}_U(1)$ on $X \wedge_U Y$ which endows it with the structure of an \mathbb{L}_U algebra. As an example, when $X = \mathbb{L}_U A$ and $Y = \mathbb{L}_U B$ are free \mathbb{L}_U -algebras,

$$(3.5) \quad X \wedge_U Y \cong \mathcal{L}_U(2)_+ \wedge (A \wedge B).$$

Analogously, we have an internal function object in $GS[\mathbb{L}_U]$ that satisfies the usual adjunction.

Definition 3.6. Let X, Y be objects of $GS[\mathbb{L}_U]$. We define the mapping \mathbb{L}_U -spectrum $F_{\mathcal{L}_U}(X, Y)$ to be the equalizer of the diagram

$$F_{GS[\mathbb{L}_U]}(\mathcal{L}_U(2)_+ \wedge X, Y) \rightrightarrows F_{GS[\mathbb{L}_U]}((\mathcal{L}_U(2) \times \mathcal{L}_U(1) \times \mathcal{L}_U(1))_+ \wedge X, Y),$$

where the maps are induced by the action of $\mathcal{L}_U(1) \times \mathcal{L}_U(1)$ on $\mathcal{L}_U(2)$ by block sum and via the adjunction homeomorphism

$$\begin{aligned} F_{GS[\mathbb{L}_U]}((\mathcal{L}_U(2) \times \mathcal{L}_U(1) \times \mathcal{L}_U(1))_+ \wedge X, Y) &\cong \\ F_{GS[\mathbb{L}_U]}((\mathcal{L}_U(2) \times \mathcal{L}_U(1))_+ \wedge X, F_{GS[\mathbb{L}_U]}(\mathcal{L}_U(1)_+ \wedge \mathbb{S}_G, Y)) \end{aligned}$$

along with the action $\mathcal{L}_U(1)_+ \wedge X \rightarrow X$ as well as the coaction

$$Y \longrightarrow F_{GS[\mathbb{L}_U]}(\mathcal{L}(1)_+ \wedge \mathbb{S}_G, Y).$$

In what follows, we will repeatedly make use of the fact that for any $n > 0$, we can choose a G -equivariant homeomorphism $U^n \rightarrow U$ and more generally we have the following lemma:

Lemma 3.7. *Let U be any G -universe. If T is a non-empty admissible set for $\mathcal{L}(U)$, then there is a G -equivariant homeomorphism*

$$\mathbb{R}\{T\} \otimes U \longrightarrow U.$$

Proof. By definition of admissibility, for the linear isometries operad we have an equivariant embedding

$$\mathbb{R}\{T\} \otimes U \longrightarrow U.$$

This implies that every isomorphism class of representations in $\mathbb{R}\{T\} \otimes U$ is contained in U . The inclusion of a trivial summand in $\mathbb{R}\{T\}$ (which exists since T is non-empty) guarantees that every irreducible representation of U is also in $\mathbb{R}\{T\} \otimes U$. \square

We now establish the basic properties of \wedge_U .

Theorem 3.8. *Let X , Y , and Z be objects of $GS[\mathcal{L}_U]$. There is a natural commutativity isomorphism*

$$\tau: X \wedge_U Y \longrightarrow Y \wedge_U X$$

and a natural associativity isomorphism

$$(X \wedge_U Y) \wedge_U Z \cong X \wedge_U (Y \wedge_U Z).$$

More generally, there is a canonical natural isomorphism

$$X_1 \wedge_U \dots \wedge_U X_k \cong \mathcal{L}_U(k) \times \underbrace{(\mathcal{L}_U(1) \times \dots \times \mathcal{L}_U(1))}_k (X_1 \wedge \dots \wedge X_k),$$

where the left-hand side is associated in any order and the right-hand side denotes the evident coequalizer generalizing the definition of \wedge_U .

Proof. Commutativity is essentially immediate (see [7, I.5.2]) and associativity is a consequence of the equivariant analogue of [7, I.5.4], that is, the isomorphism

$$\mathcal{L}_U(i+j) \cong \mathcal{L}_U(2) \times_{\mathcal{L}_U(1) \times \mathcal{L}_U(1)} \mathcal{L}_U(i) \times \mathcal{L}_U(j),$$

which holds by the same proof as nonequivariantly (using Lemma 3.7). Associativity and the last formula now follow from the arguments for [7, I.5.6]. Specifically, we have natural isomorphisms $\mathcal{L}_U(1) \times_{\mathcal{L}_U(1)} X \cong X$ for all X in $GS[\mathbb{L}_U]$, and therefore there are natural isomorphisms

$$\begin{aligned} X \wedge_U Y \wedge_U Z &\cong \mathcal{L}_U(2) \times_{\mathcal{L}_U(1) \times \mathcal{L}_U(1)} (\mathcal{L}_U(2) \times_{\mathcal{L}_U(1) \times \mathcal{L}_U(1)} (X \wedge Y)) \wedge (\mathcal{L}_U(1) \times_{\mathcal{L}_U(1)} Z) \\ &\cong (\mathcal{L}_U(2) \times_{\mathcal{L}_U(1) \times \mathcal{L}_U(1)} \mathcal{L}_U(2) \times_{\mathcal{L}_U(1)} X) \times_{\mathcal{L}_U(1) \times \mathcal{L}_U(1) \times \mathcal{L}_U(1)} (Y \wedge Z) \\ &\cong \mathcal{L}_U(3) \times_{\mathcal{L}_U(1) \times \mathcal{L}_U(1) \times \mathcal{L}_U(1)} (X \wedge Y \wedge Z). \end{aligned}$$

\square

Next, we construct the unit map, which relies on the equivariant analogue of a basic point-set property of spaces of linear isometries.

Lemma 3.9. *The orbit space $\mathcal{L}_U(2)/(\mathcal{L}_U(1) \times \mathcal{L}_U(1))$ consists of a single point. More generally, the orbit space $\mathcal{L}_U(n)/\mathcal{L}_U(1)^{\times n}$ consists of a single point.*

Proof. The right action map $\mathcal{L}_U(2) \times \mathcal{L}_U(1) \times \mathcal{L}_U(1) \rightarrow \mathcal{L}_U(2)$ is clearly a map of G -spaces. As a consequence, we can compute the orbit space as the colimit of underlying spaces, and so in this case the result follows from the non-equivariant identification of the orbit space [7, I.8.1].

We deduce the general case by induction: We can use theorem 3.8 to write

$$\mathcal{L}_U(n)/\mathcal{L}_U(1)^{\times n} \cong (\mathcal{L}_U(2) \times_{\mathcal{L}_U(1) \times \mathcal{L}_U(1)} (\mathcal{L}_U(1) \times \mathcal{L}_U(n-1))) / \mathcal{L}_U(1)^{\times n}.$$

Since coequalizers commute, the result for n now follows from the base case $n = 2$ and the induction hypothesis. \square

As a consequence, we have the following corollary:

Corollary 3.10. *There is a natural isomorphism of \mathbb{L}_U -spectra*

$$\lambda: \mathbb{S}_G \wedge_U \mathbb{S}_G \cong \mathbb{S}_G$$

such that $\lambda\tau = \lambda$.

The argument for [7, I.8.3] now generalizes without change to the equivariant setting:

Theorem 3.11. *Let X be an object of $GS[\mathbb{L}_U]$. Then there exists a natural map*

$$\psi: \mathbb{S}_G \wedge_U X \longrightarrow X$$

which is compatible with the commutativity and associativity isomorphisms.

Combining Theorems 3.8 and 3.11, we have proved the following result.

Theorem 3.12. *The category $GS[\mathbb{L}_U]$ is a closed weak symmetric monoidal category with product \wedge_U , unit \mathbb{S}_G , and function object $F_{\mathcal{L}_U}(-, -)$.*

Just as in the setting of spaces [1, 2] and spectra [7], we can actually work with the closed symmetric monoidal category obtained by restricting to the unital objects.

Definition 3.13. Let GS_U denote the full subcategory of $GS[\mathbb{L}_U]$ consisting of those objects for which $\psi: \mathbb{S}_G \wedge_U X \rightarrow X$ is an isomorphism. For X, Y objects in GS_U , let $F_U(X, Y)$ denote $\mathbb{S}_G \wedge_U F_{\mathcal{L}_U}(X, Y)$ and abusively denote by $X \wedge_U Y$ the coequalizer regarded as an object of GS_U .

Corollary 3.10 implies that there is a functor $\mathbb{S}_G \wedge_U (-): GS[\mathbb{L}_U] \rightarrow GS_U$ which is the left adjoint to the functor $F_{\mathcal{L}_U}(\mathbb{S}_G, -)$ and the right adjoint to the forgetful functor. As a consequence, we can deduce the following result.

Proposition 3.14. *The category GS_U is complete and cocomplete. Colimits are created in $GS[\mathbb{L}_U]$ (and hence in GS). Limits are formed by applying $\mathbb{S}_G \wedge_U (-)$ to the limit in $GS[\mathbb{L}_U]$. Similarly, GS_U has tensors and cotensors with based G -spaces. Tensors are created in $GS[\mathbb{L}_U]$ (and hence in GS). Cotensors are formed by applying $\mathbb{S}_G \wedge_U (-)$ to the cotensor in $GS[\mathbb{L}_U]$.*

It is now straightforward to conclude the following result.

Theorem 3.15. *The category GS_U is a closed symmetric monoidal category with unit \mathbb{S}_G , product \wedge_U , and function object $F_U(-, -)$.*

3.2. Rings and modules in $GS[\mathbb{L}_U]$ and GS_U . We now turn to the characterization of multiplicative objects in $GS[\mathbb{L}_U]$ and GS_U . The key observation about \wedge_U is that (in direct analogy with the non-equivariant case), monoids for \wedge_U are algebras over the non- Σ linear isometries operad \mathcal{L}_U and commutative monoids for \wedge_U are algebras over the linear isometries operad \mathcal{L}_U . More precisely, let \mathbb{T} and \mathbb{P} denote the monads structuring associative and commutative monoid objects in $GS[\mathbb{L}_U]$ respectively. Concretely, for X an object of $GS[\mathbb{L}_U]$,

$$\mathbb{T}X = \bigvee_{k \geq 0} \underbrace{X \wedge_U \dots \wedge_U X}_k \quad \text{and} \quad \mathbb{P}X = \bigvee_{k \geq 0} \left(\underbrace{X \wedge_U \dots \wedge_U X}_k \right) / \Sigma^k,$$

where X^0 is defined to be \mathbb{S}_G .

Monadic algebras over \mathbb{T} and \mathbb{P} in GS_U are simply algebras in $GS[\mathbb{L}_U]$ that are unital; there are functors

$$\mathbb{S}_G \wedge_U (-): (GS[\mathbb{L}_U])[\mathbb{T}] \longrightarrow GS_U[\mathbb{T}]$$

and

$$\mathbb{S}_G \wedge_U (-): (GS[\mathbb{L}_U])[\mathbb{P}] \longrightarrow GS_U[\mathbb{P}].$$

The next result connects the categories $(GS[\mathbb{L}_U])[\mathbb{T}]$ and $(GS[\mathbb{L}_U])[\mathbb{P}]$ of monadic algebras to categories of operadic \mathcal{N}_∞ algebras [3].

Theorem 3.16. *The category $(GS[\mathbb{L}_U])[\mathbb{T}]$ is isomorphic to the category of non- Σ \mathcal{L}_U -algebras in GS . The category $(GS[\mathbb{L}_U])[\mathbb{P}]$ is isomorphic to the category of \mathcal{L}_U -algebras in GS .*

Proof. The argument is the same as the proof of [7, II.4.6], using the isomorphism of equation (3.5) levelwise. \square

In light of the previous theorem, we will refer to monoids and commutative monoids in $GS[\mathbb{L}_U]$ and GS as associative and commutative \mathcal{N}_∞ ring orthogonal G -spectra, respectively.

Next, the arguments of [7, II.7] extend to prove the following:

Theorem 3.17. *The categories $(GS[\mathbb{L}_U])[\mathbb{T}]$, $(GS[\mathbb{L}_U])[\mathbb{P}]$, $GS_U[\mathbb{T}]$, and $GS_U[\mathbb{P}]$ are complete and cocomplete, with limits created in GS . The categories $(GS[\mathbb{L}_U])[\mathbb{T}]$ and $GS_U[\mathbb{T}]$ are tensored and cotensored over based G -spaces, with cotensors created in $GS[\mathbb{L}_U]$ and GS_U respectively. The categories $(GS[\mathbb{L}_U])[\mathbb{P}]$ and $GS_U[\mathbb{P}]$ are tensored and cotensored over unbased G -spaces, with cotensors created in $GS[\mathbb{L}_U]$ and GS_U respectively (regarding these categories as cotensored over unbased spaces via the functor that adjoins a disjoint G -fixed basepoint).*

As an aside, we note the following standard observation, which follows as usual simply by checking the universal property.

Lemma 3.18. *The symmetric monoidal product \wedge_U is the coproduct on $GS_U[\mathbb{P}]$.*

Finally, for any monoid or commutative monoid R , there are associated categories of (left) R -modules in $\mathbb{L}_U\mathcal{S}$ and GS_U . Since the theory is cleanest in the case of GS_U , we focus on the unital setting in the following discussion. The multiplication and unit maps for R give the functor $R \wedge_U (-)$ the structure of a monad on GS_U .

Definition 3.19. Let R be an object in $GS_U[\mathbb{T}]$ or $GS_U[\mathbb{P}]$. The category $\mathcal{M}_{R,U}$ of R -modules in $GS_U[\mathbb{P}]$ is the category of algebras for the monad $R \wedge_U (-)$ in GS_U .

Such categories of R -modules are complete and cocomplete, with limits and colimits created in GS_U . When R is commutative, the category of R -modules is closed symmetric monoidal with unit R and product $X \wedge_{R,U} Y$ defined as the coequalizer of the diagram

$$X \wedge_U R \wedge_U Y \rightrightarrows X \wedge_U Y$$

where the maps are induced by the right action of R on X via the symmetry isomorphism and the left action of R on Y . The function object is defined as the equalizer of the diagram

$$F_U(X, Y) \rightrightarrows F_U(R \wedge_U X, Y)$$

where the maps are induced by the action of R on X and the adjoint of the composite

$$R \wedge_U X \wedge_U F_U(X, Y) \longrightarrow R \wedge_U Y \longrightarrow Y.$$

There are also the evident categories of R -algebras and commutative R -algebras.

Definition 3.20. Let R be an object in $GS_U[\mathbb{P}]$. Abusively denote by \mathbb{T} and \mathbb{P} the monads in $\mathcal{M}_{R,U}$ that structure monoids and commutative monoids. We refer to the categories $\mathcal{M}_{R,U}[\mathbb{T}]$ and $\mathcal{M}_{R,U}[\mathbb{P}]$ as the categories of R -algebras and commutative R -algebras respectively.

3.3. Point-set multiplicative change of universe functors. One of the somewhat counterintuitive properties of the categories $GS[\mathbb{L}_U]$ as U varies comes from the behavior of the change of universe functors. To be clear, the change of universe functors we study in this section are *multiplicative* change of universe — the underlying (complete) additive universe that structures GS remains constant.

Let U and U' be G -universes, and denote by $\mathcal{L}(U, U')$ the G -space of non-equivariant linear isometries $U \rightarrow U'$, where G acts by conjugation. When $U = U'$, note that $\mathcal{L}(U, U) = \mathcal{L}_U(1)$.

Definition 3.21. Let U and U' be G -universes. We define the functor

$$\mathcal{L}\mathcal{I}_U^{U'} : GS[\mathbb{L}_U] \longrightarrow GS[\mathbb{L}_{U'}]$$

by setting $\mathcal{L}\mathcal{I}_U^{U'} X$ to be the coequalizer of the diagram

$$\mathcal{L}(U, U')_+ \wedge \mathcal{L}(U, U)_+ \wedge X \rightrightarrows \mathcal{L}(U, U')_+ \wedge X$$

where the maps are determined by the action of $\mathcal{L}_U(1)$ on X and the composition $\mathcal{L}(U, U') \times \mathcal{L}(U, U) \rightarrow \mathcal{L}(U, U')$. The action of $\mathcal{L}_{U'}(1)$ on $\mathcal{I}_U^{U'} X$ is also induced by the composition map $\mathcal{L}(U', U') \times \mathcal{L}(U, U') \rightarrow \mathcal{L}(U, U')$.

As explained in [6, 1.3, 1.4], we have the following point-set result about the behavior of these functors. We include the proof here in order to make this paper more self-contained.

Theorem 3.22. *Let U and U' be G -universes. The functors $\mathcal{L}\mathcal{I}_U^{U'}$ and $\mathcal{L}\mathcal{I}_{U'}^U$ are inverse equivalences of categories between $GS[\mathbb{L}_U]$ and $GS[\mathbb{L}_{U'}]$. Both functors are strong symmetric monoidal. As a consequence, the change of universe functors descend to the categories GS_U and $GS_{U'}$.*

Proof. This result follows from the identification of the coequalizer

$$\mathcal{L}(U', U'') \times \mathcal{L}(U', U') \times \mathcal{L}(U, U') \rightrightarrows \mathcal{L}(U', U'') \times \mathcal{L}(U, U')$$

as $\mathcal{L}(U, U'')$, for any universes U, U' , and U'' [6, 2.2] and where the maps are all induced by the composition γ . Since coequalizers in G -spaces are computed using the forgetful functor to spaces, it suffices to show that this is a coequalizer diagram of non-equivariant spaces. But in this setting, the diagram is a split coequalizer. The splitting is constructed as follows. Choose an isomorphism $s: U \rightarrow U'$ and define

$$h: \mathcal{L}(U, U'') \longrightarrow \mathcal{L}(U', U'') \times \mathcal{L}(U, U')$$

and

$$k: \mathcal{L}(U', U'') \times \mathcal{L}(U, U') \longrightarrow \mathcal{L}(U', U'') \times \mathcal{L}(U', U') \times \mathcal{L}(U, U')$$

via the formulas $h(f) = (f \circ s^{-1}, s)$ and $k(g', g) = (g', g \circ s^{-1}, s)$. Then $\gamma \circ h = \text{id}$, $(\text{id} \times \gamma) \circ k = \text{id}$, and $(\gamma \times \text{id}) \circ k = h \circ \gamma$. \square

3.4. Change of group and fixed-point functors. In this section, we study change-of-group and fixed-point functors in the context of the categories $GS[\mathbb{L}_U]$ and GS_U . If we are content to ignore the monoidal structure, the point-set theory of the change of group and fixed-point functors is the same as for GS . The interaction of these functors with the action of $\mathcal{L}_U(1)$ is more subtle. Our discussion relies on observations from [14, §VI.1].

Let $\iota_H: H \rightarrow G$ be the inclusion of a subgroup. Denote by WH the quotient NH/H , where NH is the normalizer of H in G . For X an object of GS , there is a homeomorphism

$$\iota_H^* \mathbb{L}_U X \cong \mathbb{L}_{(\iota_H^* U)} (\iota_H^* X).$$

This homeomorphism is easily seen to be compatible with the monad structure, and so we obtain a functor

$$\iota_H^*: GS[\mathbb{L}_U] \longrightarrow HS[\mathbb{L}_{(\iota_H^* U)}],$$

where the additive universe on HS here is ι^* applied to the complete universe structuring GS . Analogously, for Y an object of HS , we have a homeomorphism

$$G_+ \wedge_H \mathbb{L}_{(\iota_H^* U)} Y \cong \mathbb{L}_U (G_+ \wedge_H Y)$$

that is compatible with the monad structure, producing a functor

$$G_+ \wedge_H (-): HS[\mathbb{L}_{(\iota_H^* U)}] \longrightarrow GS[\mathbb{L}_U]$$

that is the left adjoint to ι_H^* . Finally, there is also a homeomorphism

$$F_H(G, \mathbb{L}_{(\iota_H^* U)}^\sharp Y) \cong \mathbb{L}^\sharp F_H(G, Y)$$

(here recall that the comonad \mathbb{L}^\sharp is described just prior to the proof of Lemma 3.3) that is compatible with the comonad structure and thus produces the right adjoint

$$F_H(G, -): HS[\mathbb{L}_{(\iota_H^* U)}] \longrightarrow GS[\mathbb{L}_U]$$

to ι_H^* .

Furthermore, all of these functors are compatible with the functors creating the unital objects, and so descend to functors $\iota_H^*: GS_U \rightarrow HS_{(\iota_H^* U)}$ and the attendant left and right adjoints.

Finally, it is evident that ι_H^* is symmetric monoidal and so it restricts to categories of monoids and commutative monoids.

Proposition 3.23. *Let H be a subgroup of G . Then there are forgetful functors*

$$\iota_H^*: (GS[\mathbb{L}_U])[\mathbb{T}] \longrightarrow (HS[\mathbb{L}_{\iota_H^* U}])[\mathbb{T}] \quad \text{and} \quad \iota_H^*: GS_U[\mathbb{T}] \longrightarrow GS_{\iota_H^* U}[\mathbb{T}]$$

and

$$\iota_H^*: (GS[\mathbb{L}_U])[\mathbb{P}] \longrightarrow (HS[\mathbb{L}_{\iota_H^* U}])[\mathbb{P}] \quad \text{and} \quad \iota_H^*: GS_U[\mathbb{P}] \longrightarrow GS_{\iota_H^* U}[\mathbb{P}].$$

Next, we turn to the question of categorical fixed-points. We work with the categorical fixed point functor $(-)^H$ on GS [14, V.3.9]. For general U , the multiplicative properties of the categorical fixed-point functor is fairly complicated. However, when U is trivial, we have the following result.

Theorem 3.24. *Let H be a subgroup of G . Then the categorical H -fixed point functor on GS descends to a lax monoidal categorical H -fixed point functor*

$$(-)^H: GS[\mathbb{L}_{\mathbb{R}^\infty}] \longrightarrow WHS[\mathbb{L}_{\mathbb{R}^\infty}]$$

which has a op-lax symmetric monoidal left adjoint

$$\epsilon_H^*: WHS[\mathbb{L}_{\mathbb{R}^\infty}] \longrightarrow GS[\mathbb{L}_{\mathbb{R}^\infty}]$$

which assigns to a WH -spectrum X the G -spectrum obtained by pulling back along the quotient $NH \rightarrow WH$, changing (additive) universe, and inducing up to G . When H is normal, the left adjoint is strong symmetric monoidal.

Proof. Since

$$(\mathcal{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)_+ \wedge X)^H \cong \mathcal{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)_+ \wedge X^H,$$

the categorical H -fixed point functor restricts to a functor

$$GS[\mathbb{L}_{\mathbb{R}^\infty}] \longrightarrow WHS_{\mathbb{R}^\infty}.$$

Analogously,

$$\epsilon_H^*(\mathcal{L}(U, U)_+ \wedge Y) \cong \mathcal{L}(U, U)_+ \wedge \epsilon_H^* Y,$$

which implies that ϵ_H^* restricts to a functor from $WHS[\mathbb{L}_{\mathbb{R}^\infty}]$ to $GS[\mathbb{L}_{\mathbb{R}^\infty}]$.

Next, we consider the interaction of $(-)^H$ with the monoidal structure. Since the action of G on $\mathcal{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ is trivial and $(-)^H$ is lax monoidal on GS [14, V.3.8], for X and Y in $GS[\mathbb{L}_{\mathbb{R}^\infty}]$ and $H \subseteq G$ we have a natural map

$$(\mathcal{L}_{\mathbb{R}^\infty}(2) \times_{\mathcal{L}_{\mathbb{R}^\infty}(1) \times \mathcal{L}_{\mathbb{R}^\infty}(1)})_+ \wedge X^H \wedge Y^H \longrightarrow (\mathcal{L}_{\mathbb{R}^\infty}(2) \times_{\mathcal{L}_{\mathbb{R}^\infty}(1) \times \mathcal{L}_{\mathbb{R}^\infty}(1)})_+ \wedge (X \wedge Y)^H$$

which lands in the fixed-points

$$((\mathcal{L}_{\mathbb{R}^\infty}(2) \times_{\mathcal{L}_{\mathbb{R}^\infty}(1) \times \mathcal{L}_{\mathbb{R}^\infty}(1)})_+ \wedge (X \wedge Y))^H,$$

and so we deduce that $(-)^H$ is a lax symmetric monoidal functor

$$GS[\mathbb{L}_{\mathbb{R}^\infty}] \longrightarrow WHS[\mathbb{L}_{\mathbb{R}^\infty}].$$

Finally, when H is normal, the left adjoint is strong symmetric monoidal since the pullback and additive change of universe are. \square

The situation for the geometric fixed point functor is analogous.

Theorem 3.25. *Let H be a subgroup of G . Then there is a lax symmetric monoidal geometric H -fixed point functor*

$$\Phi^H: GS[\mathbb{L}_{\mathbb{R}^\infty}] \longrightarrow WHS[\mathbb{L}_{\mathbb{R}^\infty}].$$

Proof. The compatibility of the geometric fixed points functor with $\mathcal{L}_U(1)$ action is clear. Next, once again the fact that the actions of H on $\mathcal{L}\mathbb{R}^\infty(2)$ and $\mathcal{L}\mathbb{R}^\infty(1)$ are trivial and the fact that Φ^H is lax symmetric monoidal on $G\mathcal{S}$ implies that it is lax symmetric monoidal on $G\mathcal{S}[\mathbb{L}_{\mathbb{R}^\infty}]$. \square

3.5. The point-set theory of the norm. In this subsection, we construct multiplicative norm functors in the sense of [10] on the categories $G\mathcal{S}[\mathbb{L}_U]$, $G\mathcal{S}_U$, and $\mathcal{M}_{R,U}$ for R a commutative algebra in $G\mathcal{S}_U$. Fix a subgroup $H \subseteq G$ and let \widehat{U} denote an H -universe. The norm functor $N_H^G: HS \rightarrow G\mathcal{S}$ is strong symmetric monoidal and so there is a natural isomorphism

$$N_H^G(\mathcal{L}_{\widehat{U}}(1)_+ \wedge X) \cong F_H(G, \mathcal{L}_{\widehat{U}}(1))_+ \wedge N_H^G X.$$

This leads to the following definition (compare with Theorem 3.8).

Definition 3.26. We define the functor

$$N_{H,\widehat{U}}^{G,U}: HS[\mathbb{L}_{\widehat{U}}] \longrightarrow G\mathcal{S}[\mathbb{L}_U]$$

on objects X via the coequalizer of the diagram

$$\mathcal{L}(\text{Ind}_H^G \widehat{U}, U)_+ \wedge F_H(G, \mathcal{L}_{\widehat{U}}(1))_+ \wedge N_H^G X \rightrightarrows \mathcal{L}(\text{Ind}_H^G \widehat{U}, U)_+ \wedge N_H^G X,$$

where the right action of U on $\mathcal{L}(\text{Ind}_H^G \widehat{U}, U)$ provides the structure of an \mathbb{L}_U algebra.

In the coequalizer, the other map is specified by the action of $F_H(G, \mathcal{L}_{\widehat{U}}(1))$ on $\mathcal{L}(\text{Ind}_H^G \widehat{U}, U)$ via the map of monoids

$$I_{(-)}: F_H(G, \mathcal{L}_{\widehat{U}}(1)) \longrightarrow \mathcal{L}_{\text{Ind}_H^G \widehat{U}}(1)$$

given by

$$f \mapsto I_f = (g \otimes u \mapsto g \otimes f(g)(u)),$$

the target of which is underlain by the orthogonal sum of isometries and hence is an isometry.

Next, we need to verify that this version of the norm functor is strong symmetric monoidal. For this purpose, we need the following technical lemmas. The first is a variant of Lemma 3.9.

Lemma 3.27. *The orbit space $\mathcal{L}(\text{Ind}_H^G \widehat{U}, U)/F_H(G, \mathcal{L}_{\widehat{U}}(1))$ consists of a single point.*

Proof. As in the proof of Lemma 3.9, since the action map

$$\mathcal{L}(\text{Ind}_H^G \widehat{U}, U) \times F_H(G, \mathcal{L}_{\widehat{U}}(1)) \longrightarrow \mathcal{L}(\text{Ind}_H^G \widehat{U}, U)$$

is a map of G -spaces, it suffices to compute the orbit space in terms of the colimit of the underlying spaces. In this case, we can deduce the result from Lemma 3.9. \square

Next, we have the following extension of [7, I.5.4].

Lemma 3.28. *Let T and T' be non-empty admissible sets for U . There are natural isomorphisms*

$$\begin{aligned} \mathcal{L}(\mathbb{R}\{T\} \otimes \widehat{U} \oplus \mathbb{R}\{T'\} \otimes \widehat{U}, U) &\cong \\ \mathcal{L}_U(2) \times_{\mathcal{L}_U(1) \times \mathcal{L}_U(1)} (\mathcal{L}(\mathbb{R}\{T\} \otimes \widehat{U}, U) \times \mathcal{L}(\mathbb{R}\{T'\} \otimes \widehat{U}, U)). \end{aligned}$$

Proof. First, observe that it suffices to show that non-equivariantly this isomorphism arises from a reflexive coequalizer diagram. Now using Lemma 3.7 to choose isomorphisms $\widehat{U} \otimes \mathbb{R}\{T'\} \cong \widehat{U}$, the required non-equivariant splittings arise just as in the proof of [7, I.5.4]. \square

The corollary of Lemma 3.28 most relevant to us here is the following:

Corollary 3.29. *There is a natural isomorphism*

$$\mathcal{L}(\mathrm{Ind}_H^G \widehat{U} \oplus \mathrm{Ind}_H^G \widehat{U}, U) \cong \mathcal{L}_U(2) \times_{\mathcal{L}_U(1) \times \mathcal{L}_U(1)} (\mathcal{L}(\mathrm{Ind}_H^G \widehat{U}, U) \times \mathcal{L}(\mathrm{Ind}_H^G \widehat{U}, U)).$$

We also have another kind of analogue of [7, I.5.4].

Lemma 3.30. *Let T be a non-empty admissible set for U . Then there is a natural isomorphism*

$$\mathcal{L}((\mathbb{R}\{T\} \otimes \widehat{U}) \oplus (\mathbb{R}\{T\} \otimes \widehat{U}), U) \cong \mathcal{L}(\mathbb{R}\{T\} \otimes \widehat{U}, U) \times_{\mathcal{L}_{\widehat{U}}(1)^T} \mathcal{L}_{\widehat{U}}(2)^T.$$

Proof. Again, the result follows by producing a reflexive coequalizer after forgetting the G -action. Specifically, we need to show that the diagram

$$\begin{array}{c} \mathcal{L}(\mathbb{R}\{T\} \otimes \widehat{U}, U) \times \mathcal{L}_{\widehat{U}}(1)^T \times \mathcal{L}_{\widehat{U}}(2)^T \\ \downarrow \downarrow \\ \mathcal{L}(\mathbb{R}\{T\} \otimes \widehat{U}, U) \times \mathcal{L}_{\widehat{U}}(2)^T \\ \downarrow \\ \mathcal{L}((\mathbb{R}\{T\} \otimes \widehat{U}) \oplus (\mathbb{R}\{T\} \otimes \widehat{U}), U) \end{array}$$

is a reflexive coequalizer. Choosing $|T|$ isomorphisms $h_i: \widehat{U}^2 \cong \widehat{U}$ such that the sum assembles to an isomorphism $h: \mathbb{R}\{T\} \otimes \widehat{U} \oplus \mathbb{R}\{T\} \otimes \widehat{U} \cong \mathbb{R}\{T\} \otimes \widehat{U}$, we can define the splitting map

$$\mathcal{L}((\mathbb{R}\{T\} \otimes \widehat{U}) \oplus (\mathbb{R}\{T\} \otimes \widehat{U}), U) \longrightarrow \mathcal{L}(\mathbb{R}\{T\} \otimes \widehat{U}, U) \times \mathcal{L}_{\widehat{U}}(2)^T$$

via $f \mapsto (f \circ h, h_1, h_2, \dots, h_{|T|})$. The argument now proceeds exactly as in [7, I.5.4]. \square

This has the following corollary.

Corollary 3.31. *For $H \subseteq G$, there is a natural isomorphism*

$$\mathcal{L}(\mathrm{Ind}_H^G \widehat{U} \oplus \mathrm{Ind}_H^G \widehat{U}, U) \cong \mathcal{L}(\mathrm{Ind}_H^G \widehat{U}, U) \times_{F_H(G, \mathcal{L}_{\widehat{U}}(1))} F_H(G, \mathcal{L}_{\widehat{U}}(2)).$$

We can now prove the following theorem.

Theorem 3.32. *The functor $N_{H, \widehat{U}}^{G, U}$ is strong symmetric monoidal.*

Proof. By Lemma 3.27, we see that $N_{H, \widehat{U}}^{G, U}$ preserves the unit. Next, we compare $N_{H, \widehat{U}}^{G, U}(X \wedge_{\widehat{U}} Y)$ and $(N_{H, \widehat{U}}^{G, U} X) \wedge_U (N_{H, \widehat{U}}^{G, U} Y)$ by direct computation. By definition, we have

$$\begin{aligned} N_{H, \widehat{U}}^{G, U}(X \wedge_{\widehat{U}} Y) = \\ \mathcal{L}(\mathrm{Ind}_H^G \widehat{U}, U) \times_{F_H(G, \mathcal{L}_{\widehat{U}}(1))} \left(N_H^G(\mathcal{L}_{\widehat{U}}(2) \times_{\mathcal{L}_{\widehat{U}}(1) \times \mathcal{L}_{\widehat{U}}(1)} (X \wedge Y)) \right). \end{aligned}$$

Since the norm functor commutes with reflexive coequalizers and is symmetric monoidal as a functor on orthogonal H -spectra, this is isomorphic to

$$\mathcal{L}(\mathrm{Ind}_H^G \widehat{U}, U) \times_{F_H(G, \mathcal{L}_{\widehat{U}}(1))} \left(F_H(G, \mathcal{L}_{\widehat{U}}(2)) \times_{F_H(G, \mathcal{L}_{\widehat{U}}(1)) \times F_H(G, \mathcal{L}_{\widehat{U}}(1))} (N_H^G X \wedge N_H^G Y) \right).$$

Applying Corollary 3.31, we rewrite this as

$$\mathcal{L}(\mathrm{Ind}_H^G \widehat{U} \oplus \mathrm{Ind}_H^G \widehat{U}, U) \times_{F_H(G, \mathcal{L}_{\widehat{U}}(1)) \times F_H(G, \mathcal{L}_{\widehat{U}}(1))} (N_H^G X \wedge N_H^G Y).$$

On the other hand, writing out $(N_{H, \widehat{U}}^{G, U} X) \wedge_U (N_{H, \widehat{U}}^{G, U} Y)$ we have

$$\mathcal{L}_U(2) \times_{\mathcal{L}_U(1) \times \mathcal{L}_U(1)} \left((\mathcal{L}(\mathrm{Ind}_H^G \widehat{U}, U) \times_{F_H(G, \mathcal{L}_{\widehat{U}}(1))} N_H^G X) \wedge (\mathcal{L}(\mathrm{Ind}_H^G \widehat{U}, U) \times_{F_H(G, \mathcal{L}_{\widehat{U}}(1))} N_H^G Y) \right).$$

Applying Corollary 3.29, we can rewrite this as

$$\mathcal{L}(\mathrm{Ind}_H^G \widehat{U} \oplus \mathrm{Ind}_H^G \widehat{U}, U) \times_{F_H(G, \mathcal{L}_{\widehat{U}}(1)) \times F_H(G, \mathcal{L}_{\widehat{U}}(1))} (N_H^G X \wedge N_H^G Y).$$

Finally, the naturality of the isomorphisms above make it clear that the pentagon identities hold. \square

As a consequence of Theorem 3.32, the functor $N_{H, \widehat{U}}^{G, U}$ restricts to a functor

$$N_{H, \widehat{U}}^{G, U} : HS_{\widehat{U}} \longrightarrow GS_U$$

which we abusively refer to with the same notation.

We now turn to establish the following adjunction on commutative ring objects. We will be predominantly interested in the case where $\widehat{U} = \iota_H^* U$, as this is relevant for describing the G -symmetric monoidal structure on the categories $\mathcal{M}_{R, U}$.

Theorem 3.33. *There are adjoint pairs with left adjoints*

$$N_{H, \iota_H^* U}^{G, U} : (HS[\mathbb{L}_{\iota_H^* U}])[\mathbb{P}] \longrightarrow (GS[\mathbb{L}_U])[\mathbb{P}] \quad \text{and} \quad N_{H, \iota_H^* U}^{G, U} : (HS_{\iota_H^* U})[\mathbb{P}] \longrightarrow GS_U[\mathbb{P}]$$

and right adjoints

$$\iota_H^* : (GS[\mathbb{L}_U])[\mathbb{P}] \longrightarrow (HS[\mathbb{L}_{\iota_H^* U}])[\mathbb{P}] \quad \text{and} \quad \iota_H^* : GS_U[\mathbb{P}] \longrightarrow HS_{\iota_H^* U}[\mathbb{P}]$$

respectively.

Proof. Our proof depends on an alternate characterization of $N_{H, \iota_H^* U}^{G, U}$ using the multiplicative change of universe functors. First, observe that the conclusion of the theorem follows immediately when $U = \mathbb{R}^\infty$: since for any G the category $GS[\mathbb{L}_{\mathbb{R}^\infty}]$ is equivalent to the category of G -objects in $\mathcal{S}[\mathbb{L}_{\mathbb{R}^\infty}]$, we can apply [10, A.56]. Next, we apply the multiplicative change of universe functors of Definition 3.21. The result is a consequence of the following identification

$$N_{H, \iota_H^* U}^{G, U} X \cong \mathcal{L} \mathcal{S}_{\mathbb{R}^\infty}^U \left(N_{H, \mathbb{R}^\infty}^{G, \mathbb{R}^\infty} (\mathcal{L} \mathcal{S}_{\iota_H^* U}^{\mathbb{R}^\infty} X) \right)$$

along with the fact that by Theorem 3.22 the change of universe functors are equivalences of categories.

To establish the identification, we expand the right-hand side, writing \mathbb{R} in place of \mathbb{R}^∞ for concision:

$$\begin{aligned}
& \mathcal{L}(\mathbb{R}, U) \times_{\mathcal{L}(\mathbb{R}, \mathbb{R})} N_{H, \mathbb{R}}^{G, \mathbb{R}} \mathcal{L}(\iota_H^* U, \mathbb{R}) \times_{\mathcal{L}(\iota_H^* U, \iota_H^* U)} X \\
& \cong \mathcal{L}(\mathbb{R}, U) \times_{\mathcal{L}_{\mathbb{R}}(1)} \left(\mathcal{L}(\text{Ind}_H^G \mathbb{R}, \mathbb{R}) \times_{F_H(G, \mathcal{L}_{\mathbb{R}}(1))} N_H^G \left(\mathcal{L}(\iota_H^* U, \mathbb{R}) \times_{\mathcal{L}_{\iota_H^* U}(1)} X \right) \right) \\
& \cong \mathcal{L}(\text{Ind}_H^G \mathbb{R}, U) \times_{F_H(G, \mathcal{L}(\mathbb{R}, \mathbb{R}))} N_H^G \mathcal{L}(\iota_H^* U, \mathbb{R}) \times_{\mathcal{L}(\iota_H^* U, \iota_H^* U)} X \\
& \cong \mathcal{L}(\text{Ind}_H^G \mathbb{R}, U) \times_{F_H(G, \mathcal{L}(\mathbb{R}, \mathbb{R}))} F_H(G, \mathcal{L}(\iota_H^* U, \mathbb{R})) \times_{F_H(G, \mathcal{L}(\iota_H^* U, \iota_H^* U))} N_H^G X \\
& \cong \mathcal{L}(\text{Ind}_H^G \iota_H^* U, U) \times_{F_H(G, \mathcal{L}(\iota_H^* U, \iota_H^* U))} N_H^G X \\
& \cong N_{H, \iota_H^* U}^{G, U} X.
\end{aligned}$$

In these expansions, note that we use the fact that the norm preserves reflexive coequalizers [10, A.54]. \square

An immediate corollary of Theorem 3.33 is that commutative ring objects have an “internal norm” map arising from the counit of the adjunction.

Corollary 3.34. *Let R be an object in $(GS[\mathbb{L}_U])[\mathbb{P}]$ or $GS_U[\mathbb{P}]$. Then there is a natural map*

$$N_{H, \iota_H^* U}^{G, U} \iota_H^* R \longrightarrow R.$$

Using the counit of the adjunction of Theorem 3.33 and the absolute norm functor described in Definition 3.26, we can express the R -relative norm for a commutative ring object R in GS_U using base-change:

Definition 3.35. Let R be an object in $GS_U[\mathbb{P}]$. We define the functor

$${}_R N_{H, \iota_H^* U}^{G, U} : \mathcal{M}_{\iota_H^* R, \iota_H^* U} \longrightarrow \mathcal{M}_{R, U}$$

via the formula

$$X \mapsto R \wedge_{N_{H, \iota_H^* U}^{G, U} R} N_{H, \iota_H^* U}^{G, U} X,$$

where the coequalizer is over the counit map and the map induced by the action of R on X .

It is clear from the definition and Theorem 3.32 that the R -relative norm is also a strong symmetric monoidal functor.

4. HOMOTOPICAL CATEGORIES OF MODULES OVER AN N_∞ ALGEBRA

In this section, we describe model structures on the categories $GS[\mathbb{L}_U]$, GS_U , and categories of algebras and modules over an algebra. The main goal of our efforts is to describe the derived functors of the norm and forgetful functors as a prelude to the construction of the G -symmetric monoidal structure.

4.1. The homotopical theory of $GS[\mathbb{L}_U]$ and GS_U . We begin by quickly reviewing some of the less commonly-used terminology from the theory of model categories that we will employ in the statements of results below. Recall from [15, 5.9] that a cofibrantly generated topological model structure is compactly generated if the domains of the generating cofibrations and acyclic cofibrations are compact and satisfy the “Cofibration Hypothesis” [15, 5.3]. Let \mathcal{C} be a complete and cocomplete topologically enriched category. An h -cofibration in \mathcal{C} is a map that is the analogue of a Hurewicz cofibration; i.e., a map $X \rightarrow Y$ such that the induced map

$Y \cup_X (X \otimes I) \rightarrow Y \otimes I$ has a retraction. The Cofibration Hypothesis for a set of maps I in a model category \mathcal{A} equipped with a forgetful functor $\mathcal{A} \rightarrow \mathcal{C}$ specifies that the following two conditions are satisfied.

- (1) For a coproduct $A \rightarrow B$ of maps in I , in any pushout

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

in \mathcal{A} , the cobase change $X \rightarrow Y$ is an h -cofibration in \mathcal{C} .

- (2) Given a sequential colimit in \mathcal{A} along maps that are h -cofibrations in \mathcal{C} , the colimit in \mathcal{A} is equal to the colimit in \mathcal{C} .

In order to be able to apply Bousfield localization, it is convenient to add the requirements that:

- (1) The domains of the generating acyclic cofibrations are small with respect to the generating cofibrations,
- (2) and the cofibrations are effective monomorphisms.

A compactly generated model category that satisfies these additional conditions is cellular [11, 12.1.1], and so admits left Bousfield localizations very generally. In mild abuse of terminology, we will use the term compactly generated to refer to a compactly generated model category that is cellular in this paper.

A model category is G -topological if it is enriched over G -spaces and satisfies the analogue of Quillen's SM7 [14, III.1.14]. There is an evident G -equivariant version of the Cofibration Hypothesis. The building block for our work in this section is the complete model structure on GS [10, B.63]. (Note that although the cited reference refers to the positive complete model structure, the existence of the complete model structure is clear.) Recall that the complete model structure has generating cofibrations given by the set of maps

$$\{G_+ \wedge_H S^{-V} \wedge S_+^{n-1} \longrightarrow G_+ \wedge_H S^{-V} \wedge D_+^n\},$$

where V varies over the (additive) universe, $n \geq 0$, and $H \subseteq G$.

Lemma 4.1. *The complete model structure is a compactly generated G -topological model structure.*

Proof. The discussion proving [10, B.63] establishes that the complete model structure is cofibrantly generated. Since the generating cofibrations in the complete model structure are h -cofibrations [10, B.64], it is straightforward to see that the complete model structure satisfies the Cofibration Hypothesis. Finally, the cofibrations are effective monomorphisms since $S_+^{n-1} \rightarrow D_+^n$ is for all $n \geq 0$, and the compactness criterion for the domain of the generating acyclics is clearly satisfied. \square

Theorem 4.2. *The category $GS[\mathbb{L}_U]$ is a compactly generated weak symmetric monoidal proper G -topological model category in which the weak equivalences and fibrations are detected by the forgetful functor $U: GS[\mathbb{L}_U] \rightarrow GS$.*

Proof. The monad \mathbb{L}_U evidently satisfies the hypotheses of (the equivariant analogue of) [15, 5.13], and so we can conclude that there is a compactly generated G -topological model structure on $GS[\mathbb{L}_U]$. The proof of the unit axiom follows

from the equivariant analogue of from [7, XI.3.1], which holds by the same proof as in the non-equivariant case. To check the monoid axiom, observe that it suffices to check on the generating (acyclic) cofibrations, and since these are obtained by $\mathcal{L}_U(1) \wedge_+ (-)$ applied to generating (acyclic) cofibrations of $G\mathcal{S}$, the result holds since it does in $G\mathcal{S}$. Finally, it is clear that $G\mathcal{S}[\mathbb{L}_U]$ is proper. \square

By construction, the adjoint pair (\mathbb{L}_U, U) is a Quillen adjunction. Since $\mathcal{L}_U(1)$ is G -contractible, we can conclude that this pair induces a Quillen equivalence between $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}$.

Proposition 4.3. *The adjoint pair (\mathbb{L}_U, U) forms a Quillen equivalence between $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}$.*

Although \mathbb{L}_U is not strong symmetric monoidal, it is close: as a consequence of Lemma 3.7, there is a homeomorphism

$$\mathbb{L}_U X \wedge_U \mathbb{L}_U Y \cong \mathbb{L}_U (X \wedge Y)$$

and more generally homeomorphisms

$$\mathbb{L}_U X_1 \wedge_U \dots \wedge_U \mathbb{L}_U X_k \cong \mathbb{L}_U (X_1 \wedge \dots \wedge X_k).$$

(The failure of \mathbb{L}_U to be strong symmetric monoidal is a consequence of the fact that these homeomorphisms ultimately depend on choices of homeomorphisms $U^k \rightarrow U$.) On the other hand, the functor $Q(-) = \mathbb{S}_G \wedge_{\Sigma_+^\infty \mathcal{L}_U(1)} (-)$ is strong symmetric monoidal [2, 4.14]. As a consequence, we have the following comparison result (where here recall that p^* denotes the right adjoint to Q which gives an object of $G\mathcal{S}$ the trivial $\mathcal{L}_U(1)$ -action).

Proposition 4.4. *The adjoint pair (Q, p^*) is a weak symmetric monoidal Quillen equivalence.*

Proof. Since p^* preserves fibrations and weak equivalences, this is clearly a Quillen adjunction. Taking $\mathbb{L}\mathbb{S}_G$ as a cofibrant replacement of the unit in $G\mathcal{S}$, we compute that $Q\mathbb{L}\mathbb{S}_G \cong \mathbb{S}_G$ and so the adjunction is monoidal. Finally, evaluation of Q on the generating cofibrations makes it clear that the natural map $QX \rightarrow UX$ is a weak equivalence for cofibrant X , and so the adjunction is a Quillen equivalence. \square

In order to retain homotopical control over $G\mathcal{S}_U$, we need to prove the equivariant analogue of [7, I.8.4, XI.2.2], i.e., that the canonical unit map $\lambda: \mathbb{S}_G \wedge_U X \rightarrow X$ is always a weak equivalence. In the proof, we make use of the following standard technical lemma:

Lemma 4.5. *Let X have a left H -action and right G -action which are compatible (i.e., X is an $H \times G$ -space). Then the coequalizer*

$$(-) \times_H X$$

specifies a functor from the category of $G' \times H$ -spaces and equivariant maps to $G' \times G$ -spaces and equivariant maps.

Proof. Let Y be a $G' \times H$ -space. It is clear that $Y \times_H X$ has a $G' \times G$ action inherited from the G' -action on Y and the G -action on X . Let $f: Y \rightarrow Y'$ be a map of $G' \times H$ -spaces. Then there is an induced map of spaces

$$\theta_f: Y \times_H X \longrightarrow Y' \times_H X$$

defined by $(y, x) \mapsto (f(y), x)$. This is a left G' -map since f is a $G' \times H$ -map; $\theta_f((g'y), x) = (f(g'y), x) = (g'f(y), x) = g'\theta_f((y, x))$. Similarly, it is a right G' -map. \square

Theorem 4.6. *For each $k > 0$, the map*

$$\gamma_k: \hat{\mathcal{L}}_U(k) = \mathcal{L}_U(2) \times_{\mathcal{L}_U(1) \times \mathcal{L}_U(1)} (\mathcal{L}_U(0) \times \mathcal{L}_U(k)) \longrightarrow \mathcal{L}_U(k)$$

induced by the operadic structure map

$$\mathcal{L}_U(2) \times \mathcal{L}_U(0) \times \mathcal{L}_U(k) \longrightarrow \mathcal{L}_U(k)$$

is a homotopy equivalence of $G \times \Sigma_k$ spaces.

Proof. First, consider the case where $k = 1$. In this case, we are considering the map

$$\gamma_1: \mathcal{L}_U(2) \times_{\mathcal{L}_U(1) \times \mathcal{L}_U(1)} (\mathcal{L}_U(0) \times \mathcal{L}_U(1)) \longrightarrow \mathcal{L}_U(1).$$

The proof of [7, XI.2.2] goes through in the equivariant context to show that γ_1 is a homotopy equivalence of G -spaces. It is helpful to decompose γ_1 as follows [14, VI.6]:

$$\mathcal{L}_U(2) \times_{\mathcal{L}_U(1) \times \mathcal{L}_U(1)} (\mathcal{L}_U(0) \times \mathcal{L}_U(1)) \xrightarrow{\theta_1} \mathcal{L}_U(2)/\mathcal{L}_U(1) \xrightarrow{\theta_2} \mathcal{L}_U(1),$$

where $\mathcal{L}_U(2)/\mathcal{L}_U(1)$ is the orbit space for the right action of $\mathcal{L}_U(1)$ on $\mathcal{L}_U(2)$ given by $(f, h) \mapsto f \circ (h \oplus \text{id})$ and equipped with the right action of $\mathcal{L}_U(1)$ specified by $([f], h) \mapsto [f \circ (\text{id} \oplus h)]$, θ_2 is the restriction to the second summand, and θ_1 is specified by $(g, 0, f) \mapsto g \circ (\text{id} \oplus f)$. Both maps are $G \times \mathcal{L}_U(1)$ maps, and θ_1 is a homeomorphism.

Now take $k > 1$. Then γ_k factors as the composite

$$\hat{\mathcal{L}}_U(k) \cong (\mathcal{L}_U(2)/\mathcal{L}_U(1)) \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k) \longrightarrow \mathcal{L}_U(1) \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k) \cong \mathcal{L}_U(k),$$

induced by γ_1 , where we are using the homeomorphism

$$\hat{\mathcal{L}}_U(k) \cong (\mathcal{L}_U(2) \times_{\mathcal{L}_U(1) \times \mathcal{L}_U(1)} (\mathcal{L}_U(0) \times \mathcal{L}_U(1))) \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k).$$

To see this, observe that $\gamma_k((g, 0, f)) = g \circ (f \oplus 0)$. On the other hand, the composite above first takes $(g, 0, f)$ to $((g \circ \text{id}), f)$, then $((g \circ \text{id}), f)$ to $(\theta_2(g), f)$, and finally $(\theta_2(g), f)$ to $\theta_2(g) \circ f = g \circ (f \oplus 0)$.

Since $\mathcal{L}_U(k)$ is a universal space for the family of subgroups of $G \times \Sigma_k$ prescribed by U , it suffices to show that $(\mathcal{L}_U(2)/\mathcal{L}_U(1)) \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k)$ is also a universal space for the same family. To do this, we will unpack part of the proof of [7, XI.2.2].

Write $U \cong U_1 \oplus U_2$ as G -spaces, where U_1 and U_2 are G -universes such that $U_1 \cong U$ and $U_2 \cong U$; we can do this by Lemma 3.7. Define $\mathcal{K}(2) \subset \mathcal{L}_U(2)$ to be $\{f \mid f(\{0\} \oplus U) \subset U_2\}$, equipped with the conjugation G -action. Next, we define

$$\hat{\mathcal{K}}_1 = \mathcal{K}(2)/\mathcal{L}_U(1)$$

and we let $\mathcal{K}_1 \subset \mathcal{L}_U(1)$ be $\{f \mid f(U) \subseteq U_2\}$ with the conjugation G -action. The map θ_2 restricts to give a G -map $\hat{\mathcal{K}}_1 \rightarrow \mathcal{K}_1$ which is compatible with the action of $\mathcal{L}_U(1)$ and so by Lemma 4.5 we have an induced $G \times \Sigma_k$ -map

$$\hat{\mathcal{K}}_1 \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k) \longrightarrow \mathcal{K}_1 \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k).$$

The non-equivariant argument in [7, XI.2.2] extends to the equivariant case to show that the map $\hat{\mathcal{K}}_1 \rightarrow \mathcal{K}_1$ is a homeomorphism of G -spaces. On the other hand, we have a homeomorphism of G -spaces $\mathcal{K}_1 \cong \mathcal{L}(U, U_2) \cong \mathcal{L}_U(1)$ which is compatible

with the action of $\mathcal{L}_U(1)$, and so Lemma 4.5 implies that there is a composite $G \times \Sigma_k$ -map

$$\mathcal{K}_1 \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k) \longrightarrow \mathcal{L}_U(1) \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k) \cong \mathcal{L}_U(k)$$

which is a homeomorphism. Putting these together, we have a $G \times \Sigma_k$ -map

$$\hat{\mathcal{K}}_1 \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k) \cong \mathcal{L}_U(k)$$

that is a homeomorphism.

To finish the argument, observe that the proof in [7, XI.2.2] extends to the equivariant context to show that the inclusion

$$\mathcal{K}(2) \longrightarrow \mathcal{L}_U(2)$$

is a G -homotopy equivalence of right $\mathcal{L}_U(1) \times \mathcal{L}_U(1)$ -spaces and therefore

$$\hat{\mathcal{K}}_1 \longrightarrow (\mathcal{L}_U(2)/\mathcal{L}_U(1))$$

is a G -homotopy equivalence of right $\mathcal{L}_U(1)$ -spaces. As a consequence, the induced map

$$\hat{\mathcal{K}}_1 \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k) \longrightarrow (\mathcal{L}_U(2)/\mathcal{L}_U(1)) \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k)$$

is a $G \times \Sigma_k$ -homotopy equivalence. \square

As a corollary of this result, we obtain the following essential theorem which follows from the outline of [7, I.8.5], using Theorem 4.6.

Theorem 4.7. *For any X in $GS[\mathbb{L}_U]$, the unit map*

$$\lambda: \mathbb{S}_G \wedge_U X \longrightarrow X$$

is a weak equivalence.

Theorem 4.7 now allows us to prove the following theorem.

Theorem 4.8. *The category GS_U is a compactly generated symmetric monoidal proper G -topological model category in which the weak equivalences are detected by the forgetful functor and the fibrations are detected by the functor $F_U(\mathbb{S}_G, -)$.*

Proof. Although $\mathbb{S}_G \wedge_U (-)$ is not a monad, the argument for [15, 5.13] again applies. As in the corresponding proof in [7, VI.4.6], consideration of the category of counital objects in $GS[\mathbb{L}_U]$ is illuminating. \square

Remark 4.9. By adjunction, a map $\mathbb{S}_G \wedge_U \mathbb{L}_U S^n \rightarrow X$ in GS_U is the same as a map $\mathbb{S}_G^n \rightarrow X$ in GS . As a consequence, the “internal” homotopy groups in GS_U determined by the free objects on spheres coincide with the homotopy groups on the underlying orthogonal G -spectrum.

The functor $\mathbb{S}_G \wedge_U (-): GS[\mathbb{L}_U] \rightarrow GS_U$ is a Quillen left adjoint and is a symmetric monoidal functor. In fact, the following proposition is straightforward to verify.

Proposition 4.10. *The adjoint pair $(\mathbb{S}_G \wedge_U (-), F_U(\mathbb{S}_G, -))$ forms a weak symmetric monoidal Quillen equivalence between $GS[\mathbb{L}_U]$ and GS_U .*

As a consequence of these results, we have the following comparison result.

Lemma 4.11. *For cofibrant $X, Y \in GS_U$ there is a natural equivalence*

$$X \wedge_U Y \longrightarrow X \wedge Y$$

and more generally for cofibrant $\{X_1, X_2, \dots, X_n\} \in GS_U$ there are natural equivalences

$$X_1 \wedge_U X_2 \wedge_U \dots \wedge_U X_n \longrightarrow X_1 \wedge X_2 \wedge \dots \wedge X_n.$$

We now turn to the study of the multiplicative structure on GS_U . The following result explains the equivariant homotopical content of the operadic smash product \wedge_U .

Theorem 4.12. *Let X be a cofibrant object of GS_U . Then there is a natural weak equivalences of $G \times \Sigma_n$ spectra*

$$(E_{\mathcal{F}_U} \Sigma_i)_+ \wedge X^{\wedge i} \simeq X^{\wedge_U i},$$

and a natural weak equivalence of G -spectra

$$(E_{\mathcal{F}_U} \Sigma_i)_+ \wedge_{\Sigma_i} X^{\wedge i} \simeq X^{\wedge_U i} / \Sigma_i,$$

where here \mathcal{F}_U denotes the family of $G \times \Sigma_n$ specified by U .

Proof. When X is free as an object of GS_U (i.e., $X = S \wedge_U \mathbb{L}_U Y$), then the result follows immediately from Theorem 3.8 and the fact that $\mathcal{L}(U^i, U) \simeq E_{\mathcal{F}_U} \Sigma_i$. The general result now follows by inductively reducing to the free case using the filtration argument of [10, B.117]. \square

In particular, Theorem 4.12 makes clear the way in which GS_U depends on the choice of U . Specifically, the $G \times \Sigma_n$ -equivariant homotopy type of the n -fold \wedge_U power of X is controlled by U , and is precisely the universal space for the family associated to $\mathcal{L}_U(n)$.

Corollary 4.13. *Let $X \rightarrow X'$ be an acyclic cofibration in GS_U . Then the induced maps*

$$\mathbb{T}X \longrightarrow \mathbb{T}X' \quad \text{and} \quad \mathbb{P}X \longrightarrow \mathbb{P}X'$$

are weak equivalences.

Corollary 4.13 provides the essential technical input for the next theorem, which is again proved using the standard outline (e.g., see [15, 5.13] or [10, B.130]).

Theorem 4.14. *The categories $GS_U[\mathbb{T}]$ and $GS_U[\mathbb{P}]$ are compactly generated proper G -topological model categories with weak equivalences and fibrations determined by the forgetful functor to GS_U .*

For a fixed ring object R , we have the following relative version of the preceding theorem.

Theorem 4.15. *For an object R in $GS_U[\mathbb{T}]$ or $GS_U[\mathbb{P}]$, the category of R -modules in GS_U is a compactly generated proper G -topological model category with weak equivalences and fibrations determined by the forgetful functor to GS_U . When R is commutative (i.e., an object in $GS_U[\mathbb{P}]$), then*

- (1) *the category of R -modules in GS_U is a compactly generated proper G -topological symmetric monoidal model category and*
- (2) *the category of R -algebras is a compactly generated proper G -topological model category.*

4.2. The homotopical theory of change of group and fixed-point functors.

In this section, we describe how to compute the derived functors of the change-of-group and fixed-point functors described in Section 3.4. Our analysis relies on the analogous theory in the setting of GS ; the following two lemmas establish that the homotopical theory for $GS[\mathbb{L}_U]$ and GS_U is tractable.

Lemma 4.16. *Let X be an object of $GS[\mathbb{L}_U]$ or GS_U . If X is cofibrant, then the underlying orthogonal G -spectrum associated to X has the homotopy type of a cofibrant object. The analogous results hold for $(GS[\mathbb{L}_U])[\mathbb{T}]$ and $GS_U[\mathbb{T}]$.*

Proof. This follows from inspection of the generating cells and the “Cofibration Hypothesis” in this context. We can assume without loss of generality that X is a cellular object. Then $X = \operatorname{colim}_n X_n$, where the colimit is sequential and along h -cofibrations. The Cofibration Hypothesis then implies that we can compute the colimit in the underlying category, and so it suffices to consider each X_n . Since each X_n is formed from X_{n-1} by attaching cells, the Cofibration Hypothesis again allows us to inductively reduce this to consideration of the generating cells, where the result is clear. \square

Lemma 4.17. *Let X be a fibrant object in $GS[\mathbb{L}_U]$, $GS[\mathbb{L}_U][\mathbb{T}]$, or $(GS[\mathbb{L}_U])[\mathbb{P}]$. Then X is fibrant in GS . Analogously, if X is fibrant in GS_U , $GS_U[\mathbb{T}]$, or $GS_U[\mathbb{P}]$, then $F_U(\mathbb{S}_G, X)$ is fibrant in GS .*

Proof. The statements about modules imply the statements about monoids and commutative monoids, as fibrations in the model structures on the categories of algebras are determined by the forgetful functors to $GS[\mathbb{L}_U]$ and GS_U respectively. The first assertion is clear for $GS[\mathbb{L}_U]$ since the fibrations are created by the forgetful functor to GS . For GS_U , the result follows from Proposition 4.10; the functor $F_U(\mathbb{S}_G, -): GS_U \rightarrow GS[\mathbb{L}_U]$ is a Quillen right adjoint. \square

The forgetful functors ι_H^* preserve all weak equivalences, and so are already derived. Their left and right adjoints can be derived by cofibrant or fibrant approximation, as a consequence of the preceding lemmas. Similarly, the (right) derived functors of the categorical fixed points can be computed by fibrant replacement and the (left) derived functors of geometric fixed points by cofibrant replacement. We summarize the situation in the following result.

Proposition 4.18.

- (1) *The forgetful functors ι_H^* preserve all weak equivalences on GS_U and $GS[\mathbb{L}_U]$.*
- (2) *The left adjoint $G_+ \wedge_H (-)$ to ι_H^* preserves weak equivalences between cofibrant objects on GS_U and $GS[\mathbb{L}_U]$. The right adjoint $F_H(G, -)$ to ι_H^* preserves weak equivalences between fibrant objects on GS_U and $GS[\mathbb{L}_U]$.*
- (3) *The categorical fixed point functor $(-)^H$ preserves weak equivalences between fibrant objects in GS_U and $GS[\mathbb{L}_U]$.*
- (4) *The geometric fixed point functor Φ^H preserves weak equivalences between cofibrant objects in GS_U and $GS[\mathbb{L}_U]$.*

Finally, we have the following result which shows that the geometric fixed-point functor is strong monoidal in the homotopical sense.

Proposition 4.19. *Let X and Y be cofibrant objects in $GS[\mathbb{R}^\infty]$ or $GS_{\mathbb{R}^\infty}$. Then the natural map*

$$\Phi^H X \wedge_{\mathbb{R}^\infty} \Phi^H Y \longrightarrow \Phi^H (X \wedge_{\mathbb{R}^\infty} Y)$$

is a weak equivalence.

Proof. The result follows from the result for Φ^H on GS [14, V.4.7] when X and Y are generating cells, since

$$\mathbb{L}_{\mathbb{R}^\infty} X' \wedge_{\mathbb{R}^\infty} \mathbb{L}_{\mathbb{R}^\infty} Y' \cong \mathcal{L}_{\mathbb{R}^\infty}(2)_+ \wedge (X' \wedge Y')$$

for any X' and Y' . Since $(-) \wedge_{\mathbb{R}^\infty} (-)$ preserves colimits in either variable and preserves weak equivalences between cofibrant objects, we can conclude the general statement. \square

4.3. The homotopical theory of the norm. In this section, we show that the norm $N_{H, \iota_H^* U}^{G, U}$ is a homotopical functor and participates in a Quillen adjunction when restricted to commutative ring objects.

Theorem 4.20. *Let X be a cofibrant object in $GS[\mathbb{L}_U]$ or GS_G . The natural map*

$$N_{H, \iota_H^* U}^{G, U} X \longrightarrow N_H^G X$$

is a weak equivalence when G/H is admissible for U .

Proof. By induction over the cellular filtration, it suffices to consider the case when X is free. In this case, we're looking at the map

$$\mathcal{L}(\text{Ind}_H^G \iota_H^* U, U)_+ \wedge N_H^G X \longrightarrow N_H^G X$$

given by the collapse map $\mathcal{L}(\text{Ind}_H^G \iota_H^* U, U)_+ \rightarrow S^0$. Since the collapse is a G -equivalence when G/H is admissible, the result follows. \square

Remark 4.21. When G/H is not admissible for U , it is not clear in general what the homotopy type of $N_{H, \iota_H^* U}^{G, U}$ is. For free objects, the homotopy type is controlled by $\mathcal{L}(\text{Ind}_H^G \iota_H^* U, U)$, which has no G -fixed points.

Corollary 4.22. *The functor $N_{H, \iota_H^* U}^{G, U}$ preserves weak equivalences between cofibrant objects in $HS_{\iota_H^* U}$ and $HS[\mathbb{L}_{\iota_H^* U}]$ when G/H is admissible for U .*

The next lemma provides homotopical control on the output of the norm functor.

Lemma 4.23. *Let X be a cofibrant object in $HS[\mathbb{L}_{\widehat{U}}]$ or $HS_{\widehat{U}}$. Then $N_{H, \iota_H^* U}^{G, U} X$ is cofibrant in $GS[\mathbb{L}_U]$.*

Proof. Using the filtration of [10, A.3.4], we can inductively reduce to the case when X is of the form $\mathcal{L}_{\widehat{U}}(1)_+ \wedge Y$. In this case,

$$N_{H, \iota_H^* U}^{G, U}(\mathcal{L}_{\widehat{U}}(1)_+ \wedge Y) \cong \mathcal{L}(\text{Ind}_H^G \widehat{U}, U)_+ \wedge N_H^G Y.$$

Since $\mathcal{L}(\text{Ind}_H^G \widehat{U}, U) \cong \mathcal{L}_1(U)$ by Lemma 3.7, the result follows. \square

As a consequence, we have the following result about the composition of the norm functor.

Proposition 4.24. *Fix $H_1 \subseteq H_2 \subseteq G$. Let X be a cofibrant object in $H_1 S[\mathbb{L}_{\iota_{H_1}^* U}]$ or $H_1 S_{\iota_{H_1}^* U}$. Then there is a natural weak equivalence*

$$N_{H_1, \iota_{H_1}^* U}^{G, U} X \simeq N_{H_2, \iota_{H_2}^* U}^{G, U} N_{H_1, \iota_{H_1}^* U}^{H_2, \iota_{H_2}^* U} X.$$

Proof. Expanding using the definition, we have

$$N_{H_2, \iota_{H_2}^* U}^{G, U} N_{H_1, \iota_{H_1}^* U}^{H_2, \iota_{H_2}^* U} X = \mathcal{L}(\text{Ind}_{H_2}^G \iota_{H_2}^* U, U) \times_{F_{H_2}(G, \mathcal{L}_{\iota_{H_2}^* U}(1))} N_{H_2}^G \left(N_{H_1, \iota_{H_1}^* U}^{H_2, \iota_{H_2}^* U} X \right)$$

and

$$N_{H_1, \iota_{H_1}^* U}^{H_2, \iota_{H_2}^* U} X = \left(\mathcal{L}(\text{Ind}_{H_1}^{H_2} \iota_{H_1}^* U, \iota_{H_2}^* U) \times_{F_{H_1}(H_2, \mathcal{L}_{\iota_{H_1}^* U}(1))} N_{H_1}^{H_2} X \right)$$

which implies that

$$N_{H_2}^G \left(N_{H_1, \iota_{H_1}^* U}^{H_2, \iota_{H_2}^* U} X \right) \cong \left(F_{H_2}(G, \mathcal{L}(\text{Ind}_{H_1}^{H_2} \iota_{H_1}^* U, \iota_{H_2}^* U)) \times_{F_{H_2}(G, \mathcal{L}_{\iota_{H_1}^* U}(1))} N_{H_1}^G X \right).$$

Next, we show that

$$\mathcal{L}(\text{Ind}_{H_2}^G \iota_{H_2}^* U, U) \times_{F_{H_2}(G, \mathcal{L}_{\iota_{H_2}^* U}(1))} F_{H_2}(G, \mathcal{L}(\text{Ind}_{H_1}^{H_2} \iota_{H_1}^* U, \iota_{H_2}^* U))$$

is isomorphic to $\mathcal{L}(\text{Ind}_{H_1}^G \iota_{H_1}^* U, U)$. There is an equivariant map

$$\mathcal{L}(\text{Ind}_{H_2}^G \iota_{H_2}^* U, U) \times_{F_{H_2}(G, \mathcal{L}(\text{Ind}_{H_1}^{H_2} \iota_{H_1}^* U, \iota_{H_2}^* U))} \longrightarrow \mathcal{L}(\text{Ind}_{H_1}^G \iota_{H_1}^* U, U)$$

induced by composition and the natural map

$$F_{H_2}(G, \mathcal{L}(\text{Ind}_{H_1}^{H_2} \iota_{H_1}^* U, \iota_{H_2}^* U)) \longrightarrow \mathcal{L}(\text{Ind}_{H_1}^G \iota_{H_1}^* U, \text{Ind}_{H_2}^G \iota_{H_2}^* U)$$

induced by the direct sum. This map is compatible with the maps determining the coequalizer, and so it suffices to check that the underlying non-equivariant diagram is a reflexive coequalizer. This now follows from Lemma 3.30. The theorem is now a consequence of the preceding isomorphism and Lemma 4.23. \square

In the case of commutative monoid objects, it is straightforward to check that the adjunction involving the norm and the forgetful functor is homotopical; it is clear that ι_H^* preserves fibrations and weak equivalences.

Theorem 4.25. *The adjoint pairs*

$$N_{H, \iota_H^* U}^{G, U} : (HS[\mathbb{L}_{\iota_H^* U}])[\mathbb{P}] \rightleftarrows (GS[\mathbb{L}_U])[\mathbb{P}] : \iota_H^*$$

and

$$N_{H, \iota_H^* U}^{G, U} : HS_{\iota_H^* U}[\mathbb{P}] \rightleftarrows GS_U[\mathbb{P}] : \iota_H^*$$

are Quillen adjunction.

Note however that the derived functor of the norm $N_{H, \iota_H^* U}^{G, U}$ on commutative rings only agrees with the derived functor of the module norm when G/H is admissible for U ; the following result is a consequence of the fact that derived functor of the norm on commutative rings in orthogonal H -spectra agrees with the underlying norm [10, B.148].

Proposition 4.26. *Let X be a cofibrant object in $GS[\mathbb{L}_U][\mathbb{P}]$. The natural map*

$$N_{H, \iota_H^* U}^{G, U} X \longrightarrow N_H^G X$$

is a weak equivalence when G/H is admissible for U .

We now turn to the relative norm construction.

Theorem 4.27. *The functor ${}_R N_{H, \iota_H^* U}^{G, U}$ preserves weak equivalence between cofibrant objects in $\mathcal{M}_{\iota_H^* R, \iota_H^* U}$ when G/H is admissible for U .*

Proof. Since the R -relative norm is strong symmetric monoidal, it suffices to show that when X is cofibrant in $\mathcal{M}_{\iota_H^* R, \iota_H^* U}$, $N_{H, \iota_H^* U}^{G, U} X$ is cofibrant as an $N_{H, \iota_H^* U}^{G, U} R$ module. Once again, it suffices to check this on free objects, where it is straightforward. \square

5. G -SYMMETRIC MONOIDAL CATEGORIES OF MODULES OVER AN \mathcal{N}_∞ ALGEBRA

In this section, we describe the homotopical G -symmetric monoidal structure on $\mathcal{M}_{R, U}$. We characterize this structure in terms of a homotopical exponential functor

$$N^T: \mathcal{M}_{R, U} \longrightarrow \mathcal{M}_{R, U}$$

for any G -set T . We explain how this “internal norm” arises from structure on the collection of norms and forgetful functors on the categories $\mathcal{M}_{\iota_H^* R, \iota_H^* U}$ as H varies over the closed subgroups of G ; these functors assemble into an incomplete Mackey functor in homotopical categories. We also explain the resulting structure on commutative monoid objects, recovering the characterizations of [3, 6.11].

5.1. The G -symmetric monoidal structure on GS and \mathcal{M}_R . In this subsection, we review the canonical G -symmetric monoidal structure on GS and \mathcal{M}_R for R a commutative ring orthogonal G -spectrum. We begin by recalling from [3, §6] the definition of the internal norm in orthogonal spectra.

Definition 5.1. Let $H \subset G$ be a closed subgroup. The internal norm of an orthogonal G -spectrum X is specified by the formula

$$N^{G/H} X = N_{H, \iota_H^*}^G X.$$

For an arbitrary G -set T , we define the internal norm by decomposing T into a disjoint union of orbits $\coprod_i G/H_i$ and defining

$$N^T X = \bigwedge_i N^{G/H_i} X.$$

For example, when T is a trivial G -set, $N^T M$ is simply the smash-power of $|T|$ copies of M . Note that this definition extends in the evident way to categories of modules over a commutative ring orthogonal G -spectrum.

There is another equivalent description for this which will make the properties of the norm (summarized in Theorem 5.5 below) more transparent. If T is a finite G -set, then let $B_T G$ denote the translation category of T . This has object set T itself and the morphism set is $T \times G$ with structure maps the projection onto T and the action. Given a G -spectrum X , we have a $B_T G$ -shaped diagram X^T described by $t \mapsto X$ and (t, g) acts as multiplication by g on X .

A map of finite G -sets $f: T \rightarrow S$ produces a covering category $B_T G \rightarrow B_S G$ as in [10, Definition A.24], and therefore we have an associated indexed monoidal product f_*^\otimes .

Proposition 5.2. *Let $p: T \rightarrow *$ be the terminal map. There is a canonical isomorphism*

$$p_*^\otimes X^T \cong N^T(X).$$

Proof. Both sides take disjoint unions to smash products (the left by construction and the right by definition), so it suffices to show this for $T = G/H$. Both sides are then indexed products. We now unpack both sides.

Since this is a point-set statement, we can work in the equivalent category of orthogonal spectra with a G -action and prove the desired equality there. In this case, both sides are the indexed product associated to $p: G/H \rightarrow *$, so it will suffice to show that the resulting diagrams are isomorphic. For the left-hand side, the diagram is the constant diagram $X^{G/H}$. For the right-hand side, the diagram is determined by choosing coset representatives and sending a coset gH to the gHg^{-1} -spectrum $g \cdot i_H^* X$ (where here $g \cdot Y$ for an H -spectrum Y is just the restriction along the isomorphism $gHg^{-1} \cong H$). However, we then have an equivariant isomorphism of diagrams

$$X^{G/H} \cong (gH \mapsto g \cdot i_H^* X)$$

which at a coset gH is simply multiplication by g . The indexed products are therefore isomorphic. \square

Remark 5.3. The key step in the argument is the same as the one showing that we have canonical isomorphisms

$$F_H(G_+, i_H^* X) \cong F(G/H_+, X) \text{ and } G_+ \wedge_H i_H^* X \cong G/H_+ \wedge X.$$

In each case, we have the same two diagrams as the one given above and then we compare the associated indexed monoidal products.

Because N_H^G , $- \wedge -$, and i_H^* preserve weak equivalences and cofibrant objects, the internal norm is a homotopical functor.

Lemma 5.4. *For any G -set T , the internal norm N^T preserves acyclic cofibrations.*

We can now recall the basic theorem establishing the G -symmetric monoidal structure on G -spectra. All of this follows easily from Appendix A of [10]; for convenience, we include details here.

Theorem 5.5.

- (1) *For $H_1 \subseteq H_2 \subseteq G$, there is a natural isomorphism*

$$\iota_{H_1}^* \cong \iota_{H_1}^* \iota_{H_2}^*.$$

- (2) *For G -sets T_1 and T_2 , there is a natural isomorphism*

$$N^{T_1 \times T_2} X \simeq N^{T_1} N^{T_2} X.$$

- (3) *For $K \subset H$, there is a natural isomorphism*

$$\iota_K^* N^T X \simeq N^{\iota_K^* T} \iota_K^* X$$

Proof. The first part is obvious. For the second and third parts, we use the alternative description of $N^T X$ given by Proposition 5.2.

For the second, observe that $B_{T_1 \times T_2} G \cong B_{T_1} G \times B_{T_2} G$, and the composite of the norms is the composites of the indexed products

$$T_1 \times T_2 \longrightarrow T_1 \longrightarrow *.$$

The composite of the indexed products is the indexed product of the composites [10, Prop A.29].

The third is the variant of the double coset formula here. If T is a finite G -set, then we have a pullback diagram of categories

$$\begin{array}{ccc} B_{G/H \times T} G & \longrightarrow & B_T G \\ \downarrow & & \downarrow \\ B_{G/H} G & \longrightarrow & BG. \end{array}$$

Since $G/H \times T \cong G \times_H i_H^* T$, the left-hand side of this diagram is equivalent to $B_{i_H^* T} H \rightarrow BH$. Since the map on spectra induced by pulling back along $B_{G/H} G \rightarrow BG$ is i_H^* , we conclude by [10, Prop A.31] that $i_H^* N^T X \cong N^{i_H^* T} i_H^* X$. \square

The analogue of Theorem 5.5 for modules over a commutative ring orthogonal G -spectrum R follows from the characterization of the R -relative norm via the formula

$${}_R N_H^G X \cong R \wedge_{N_H^G R} N_H^G X$$

and the fact that the norm N_H^G is the left adjoint to the restriction functor ι_H^* on commutative rings. We explain in detail the argument below in the proof of Theorem 5.10.

5.2. The G -symmetric monoidal structure on GS_U and $\mathcal{M}_{R,U}$. We now provide the analogous definitions in our context.

Definition 5.6. Given $H \subset G$ a closed subset, we define the internal norm

$${}_R N_U^{G/H} M : \mathcal{M}_{R,U} \longrightarrow \mathcal{M}_{R,U}$$

as the composite

$${}_R N_U^{G/H} (-) := {}_R N_{H, \iota_H^*}^{G,U} \iota_H^* (-).$$

We extend the internal norm to an arbitrary G -set T by decomposing T into a disjoint union of orbits $\coprod_i G/H_i$ and specifying that

$${}_R N_U^T M = \bigwedge_i {}_R N^{G/H_i} M.$$

We now describe the homotopical properties of the internal norm. We begin by considering the absolute case where $R = S$.

Lemma 5.7. *Let T be an admissible G -set. The functor N^T preserves weak equivalences between cofibrant objects.*

Proof. This is a consequence of the fact that ι_H^* preserves cofibrant orthogonal G -spectra [14, V.2.2], colimits, and the identification

$$\iota_H^* \mathcal{L}_U(1)_+ \wedge X \cong \mathcal{L}_{\iota_H^* U}(1)_+ \wedge (\iota_H^* X).$$

\square

Furthermore, we can also identify the interaction of N^T with the cartesian product.

Lemma 5.8. *When T_1 and T_2 are admissible G -sets and M is a cofibrant object in $GS[\mathbb{L}_U]$, there is a natural weak equivalence*

$$N^{T_1 \times T_2} M \simeq N^{T_1}(N^{T_2} M).$$

Proof. This follows from Lemma 5.7, Lemma 5.4, and Theorem 4.20. \square

Proposition 4.24 shows that the norm functors compose as expected, and it is clear that for $H_1 \subseteq H_2 \subseteq G$, $\iota_{H_1}^* \cong \iota_{H_1}^* \iota_{H_2}^*$.

Theorem 5.9. *Fix $K \subseteq H$, let T be an admissible H -set, and let M be a cofibrant object in $GS[\mathbb{L}_U]$. The composite $\iota_K^* N^T M$ is naturally equivalent to $N^{\iota_K^* T} \iota_K^* M$.*

Proof. This again follows from Theorem 4.20 and the fact that the desired equivalence holds for the norm in orthogonal spectra. \square

When R is no longer necessarily the sphere, we have corresponding analogues of the preceding results; we summarize the situation in the following theorem.

Theorem 5.10. *Let R be a cofibrant object in $GS_U[\mathbb{P}]$.*

- (1) *The functor ${}_R N_U^T$ preserves weak equivalences between cofibrant objects.*
- (2) *When T_1 and T_2 are admissible G -sets and M is a cofibrant object in $\mathcal{M}_{R,U}$, there is a natural equivalence*

$${}_R N_U^{T_1 \times T_2} M \simeq {}_R N_U^{T_1} ({}_R N_U^{T_2} M).$$

- (3) *For $H_1 \subseteq H_2 \subseteq G$, there is a natural isomorphism*

$$\iota_{H_1}^* \cong \iota_{H_1}^* \iota_{H_2}^*.$$

- (4) *For $K \subset H$ and T an admissible G -set, there is a natural equivalence*

$$\iota_K^* {}_R N_U^T M \simeq {}_R N_U^{\iota_K^* T} \iota_K^* M$$

when M is a cofibrant object in $\mathcal{M}_{R,U}$.

Proof. The first of these follows from Theorem 4.27. The second is a consequence of Theorem 4.20; the proof is analogous to the proof of Lemma 5.8, along with the observation that the smash product defining the relative norm computes the derived smash product under our hypotheses. The third is immediate. For the fourth, we can leverage the absolute result as follows.

Since ι_K^* is a strong symmetric monoidal functor, we have the isomorphisms

$$\iota_K^* {}_R N_U^T M \cong \iota_K^* \left(N_U^T M \wedge_{N_U^T R} R \right) \cong (\iota_K^* N_U^T M) \wedge_{\iota_K^* N_U^T R} \iota_K^* R.$$

By Theorem 5.9, we know that

$$\iota_K^* N_U^T M \simeq N^{\iota_K^* T} \iota_K^* M.$$

Moreover, since N_U^T is a left adjoint on commutative rings, we have an isomorphism

$$\iota_K^* N_U^T R \cong N_{\iota_H^* U}^{\iota_H^* T} \iota_H^* R,$$

which is compatible with the counit $N_U^T R \rightarrow R$ used in the formation of the relative smash product. Since the hypotheses guarantee we are computing the derived smash product, we end up with a natural weak equivalence

$$\iota_K^* {}_R N_U^T M \simeq N^{\iota_K^* T} \iota_K^* M \wedge_{N_{\iota_H^* U}^{\iota_H^* T} \iota_H^* R} \iota_H^* R \cong {}_R N_U^{\iota_K^* T} \iota_K^* M.$$

\square

5.3. The multiplicative structure on N_∞ algebras. In this subsection, we explain how the G -symmetric monoidal structure on $G\mathcal{S}_U$ induces additional multiplicative structure on objects of $G\mathcal{S}_U[\mathbb{P}]$. Of course, Theorem 3.16 implies that an object of $G\mathcal{S}_U[\mathbb{P}]$ is an N_∞ algebra structured by the equivariant linear isometries operad determined by U , and [3, 6.11] explains the extra structure this gives. Our purpose here is to demonstrate that this structure is essentially an immediate consequence of Theorem 5.10.

Let R be a cofibrant object of $G\mathcal{S}_U[\mathbb{P}]$. The adjunction of Theorem 4.25 yields homotopical counit maps

$$N_U^{G/H} = N_{H, \iota_H^*}^G \iota_H^* R \longrightarrow R$$

for admissible G/H , which clearly induce natural maps

$$N_U^T R \longrightarrow R \quad \text{and} \quad G_+ \wedge_H N_U^S \iota_K^* R \longrightarrow R,$$

for admissible G -sets T and admissible $K \subseteq G$ sets S . The argument of [3, 6.8] extends without change to produce a map

$$N_U^T R \longrightarrow N_U^S R$$

given any G -map $f: S \rightarrow T$.

It is clear from the definition of N_U^T that the diagram

$$\begin{array}{ccc} N_U^S \amalg^T R \cong N_U^S R \wedge N_U^T R & \longrightarrow & R \wedge R \\ \downarrow & \nearrow & \\ R & & \end{array}$$

commutes. Assertion (2) of Theorem 5.10 implies that the diagram

$$\begin{array}{ccc} N_U^{S \times T} R \cong N_U^S N_U^T R & \longrightarrow & N_U^T R \\ \downarrow & \nearrow & \\ R & & \end{array}$$

commutes. Finally, assertion (4) of Theorem 5.10 implies that for any admissible sets S and T such that for some $K \subseteq G$ we have $\iota_K^* S \cong \iota_K^* T$, the diagram

$$\begin{array}{ccc} \iota_K^* N_U^S R \cong N_{\iota_K^* U}^{\iota_K^* S} \iota_K^* R & \xrightarrow{\cong} & N_{\iota_K^* U}^{\iota_K^* T} \iota_K^* R \cong \iota_K^* N_U^T R \\ & \searrow & \swarrow \\ & R & \end{array}$$

commutes. Thus, we precisely recover the characterizations of [3, 6.11].

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